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Crack paths in three-dimensional elastic solids. II: three-term expansion of the stress intensity factors—applications and perspectives

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Abstract

This work continues the calculation of the stress intensity factors, as a function of position s along the front of an arbitrary (kinked and curved) infinitesimal extension of some arbitrary crack on some three-dimensional body. More precisely, ε denoting a small parameter which the crack extension length is proportional to, what is studied here is the third term, proportional to $\varepsilon^1 = \varepsilon$ and noted $\mathbf{K}^{(1)}(s)\varepsilon$, of the expansion of these stress intensity factors at the point s of the crack front in powers of ε. The novelties with respect to previous works due to Gao and Rice on the one hand and Nazarov on the other hand, are that both the original crack and its extension need not necessarily be planar, and that a kink (discontinuity of the tangent plane to the crack) can occur all along the original crack front. Two expressions of $\mathbf{K}^{(1)}(s)$ are obtained; the difference is that the first one is more synthetic whereas the second one makes the influence of the kink angle (which can vary along the original crack front) more explicit. Application of some criterion then allows to obtain the *a priori* unknown geometric parameters of the small crack extension (length, kink angle, curvature parameters). The small scale "segmentation" of the crack front which is observed experimentally in the presence of mode III is disregarded here because a large scale point of view is adopted; this phenomenon will be discussed in a separate paper. It is shown how these results can be used to numerically predict crack paths over arbitrary distances in three dimensions. Simple applications to problems of configurational stability and bifurcation of the crack front are finally presented. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

We refer to Sections 1 and 2 of Part I for a statement of the problem examined and the notations used. It was shown in Part I that in the most general three-dimensional case, the expressions (8)

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and (19) of the first two terms, $\mathbf{K}^*(s)$ and $\mathbf{K}^{(1/2)}(s)\sqrt{\varepsilon\eta(s)} = \mathbf{K}^{(1/2)}(s)\sqrt{\delta(s)}$, of the expansion of the stress intensity factors (SIFs) in powers of some small time-like parameter ε, appeared as natural extensions of those corresponding to the two-dimensional case. In particular, both the crack extension length $\delta(s') \equiv \varepsilon\eta(s')$ and the kink angle $\varphi(s')$, which depended upon the current position s' along the crack front, appeared only through their *local* values at that point s of the front where the variation of the SIFs was to be evaluated.

In contrast, the study of the third term, $\mathbf{K}^{(1)}(s)\varepsilon$, of the expansion of the SIFs will evidence some dependence upon the values of the functions $\eta(s')$ and $\varphi(s')$ *all along the crack front*, through some integral term. This feature will considerably complicate the treatment; in order for its directrix to remain understandable, many of its technical details will be relegated to Appendices.

In fact, the "non-local" character of the third term of the expansion of the SIFs (that is, the fact that it involves an integral over the whole crack front) was already apparent in the works of Rice (1985), Gao and Rice (1986, 1987a, b), Gao (1988), Rice (1989) and Nazarov (1989). Some detailed comparison with the results of these authors will be provided. However, it is worth stressing at once that the essential originality of our approach lies in the possibility of an arbitrary geometry of the crack and its extension, and in particular of the existence of some kink angle between the original crack and its extension, whereas all works quoted above considered only *coplanar extensions of an initially plane crack*. Accounting for non-coplanarity is essential to deal with mixed mode situations.

Our essential results are contained in eqns (30) [or (30′)] (in Section 4) and (32) (in Section 5) of the text. The first one evidences the influence of the "propagation rate" of the crack front $\eta(s')$ upon $\mathbf{K}^{(1)}(s)$. Indeed it splits this quantity into a term proportional to $\eta(s)$ (representing its value for a *uniform* propagation rate: $\eta(s') \equiv \eta(s)$, $\forall s'$), another one proportional to the derivative $\eta'(s)$ of $\eta(s)$ along the crack front, and a last one which is an integral over the front involving the "fluctuations" $\eta(s') - \eta(s)$ of the propagation rate. This expression will be sufficient for the applications envisaged at the end. However, it does not clearly display the influence of the function $\varphi(s')$ upon $\mathbf{K}^{(1)}(s)$, which remains implicit in some terms. Formula (32) makes it more explicit, at the expense of a greater complexity, by again distinguishing between the influences of the local kink angle $\varphi(s)$, its derivative $\varphi'(s)$ along the front and the fluctuations of the function $\varphi(s')$. That second formula will be necessary for the theoretical study of small-scale crack front "segmentation" in mixed mode I+III, to be carried out in some future paper.

The final aim of the present paper is achieved in Section 7. It consists in combining the expansion of the SIFs in powers of the crack extension length with some appropriate propagation criterion in order to predict the propagation path. The criterion adopted here itself combines the "principle of local symmetry" of Goldstein and Salganik (1974) and some Griffith-type energetic criterion; as explained in the text, this means disregarding the "segmentation" of the crack front arising from the presence of mode III. Two cases are distinguished according to whether the crack extension is kinked or not. However, in both cases it reveals possible to *separately* determine the length of the crack extension and its kink angle and curvature parameters. Potential applications of the formulae obtained to fully three-dimensional numerical predictions of crack paths over arbitrarily long distances are presented. Also, as another, simple application, we consider in Section 8 the problem of configurational stability of the crack front vs small in-plane perturbations, for semi-infinite and penny-shaped cracks loaded in Mode I+III. These problems were previously studied by Gao and Rice (1986) and Gao (1988), assuming propagation to be strictly coplanar. It is shown that introduction of the curvature parameters of the crack extension does not modify Gao and Rice's conclusions in any way.

2. Preliminaries on $\mathbf{K}^{(1)}(s)$

Equation (15) of Part I gave the two-term expansion of the functional $\mathscr{L}$ in powers of ε. We now introduce its three-term expansion:[1]

$$\mathscr{L}[R,\mathbf{C},\Gamma,\varphi,\varphi',a^*,C^*,\varepsilon\eta,\eta'/\eta;\mathscr{T}] = \mathscr{L}^*[R,\mathbf{C},\Gamma,\varphi;\mathscr{T}]+\mathscr{L}^{(1/2)}[R,\mathbf{C},\Gamma,\varphi,a^*;\mathscr{T}]\sqrt{\varepsilon\eta}$$
$$+\mathscr{L}^{(1)}[R,\mathbf{C},\Gamma,\varphi,\varphi',a^*,C^*,\eta'/\eta;\mathscr{T}]\varepsilon\eta+O(\varepsilon^{3/2}). \quad (1)$$

The arguments φ', C^* and η'/η have now been discarded in the function $\mathscr{L}^{(1/2)}$; this is because $\mathbf{K}^{(1/2)}$, and therefore $\mathscr{L}^{(1/2)}$, have been proved in Section 6 of Part I to depend on the geometric parameters of the crack extension only through the arguments φ (kink angle) and a^* (first "curvature parameter").[2] The functional $\mathscr{L}^{(1)}$, just as $\mathscr{L}^*$ and $\mathscr{L}^{(1/2)}$, is linear in $\mathscr{T}$ and indefinitely differentiable with respect to all its geometric arguments, since it does not depend upon the extension length $\varepsilon\eta$.

Also, eqn (13) of Part I implies that the traction field $\mathscr{T}(R,\varepsilon)$ exerted on the boundary of the sphere of centre s and radius R, as a result of the application of the prescribed displacements and tractions $\mathbf{u}^p$, $\mathbf{t}^p$ on $\partial\Omega_u$ and $\partial\Omega_t$, admits an expansion in powers of ε of the form

$$\mathscr{T}(R,\varepsilon) = \mathscr{T}(R)+\mathscr{T}^{(1)}(R)\varepsilon+O(\varepsilon^{3/2}), \quad (2)$$

without any term proportional to $\varepsilon^{1/2}$.

Combining eqn (6) of Part I with eqns (1) and (2), one gets

$$\mathbf{K}(\varepsilon) = \mathbf{K}^*+\mathbf{K}^{(1/2)}\sqrt{\varepsilon\eta}+\mathbf{K}^{(1)}\varepsilon+O(\varepsilon^{3/2}) \quad (3)$$

where $\mathbf{K}^*$ and $\mathbf{K}^{(1/2)}$ are given by eqns (7) and (18) of Part I, and $\mathbf{K}^{(1)}$ by

$$\mathbf{K}^{(1)} = \mathscr{L}^{(1)}[R,\mathbf{C},\Gamma,\varphi,\varphi',a^*,C^*,\eta'/\eta;\mathscr{T}(R)]\eta+\mathscr{L}^*[R,\mathbf{C},\Gamma,\varphi;\mathscr{T}^{(1)}(R)]. \quad (4)$$

The novelty here with respect to the expressions of $\mathbf{K}^*$ and $\mathbf{K}^{(1/2)}$ is that $\mathbf{K}^{(1)}$ depends on the stress field *after the kink* through its first derivative with respect to ε. This is bound to be a source of trouble, all the more so since eqn (13) of Part I shows that the expression of this derivative consists of an integral over the whole crack front, which will result in an expression of "non-local" nature for $\mathbf{K}^{(1)}$.

Now, using eqn (1) and expanding eqn (4) of Part I to order $\varepsilon^1 \equiv \varepsilon$ instead of just $\sqrt{\varepsilon}$ as we did in Section 6 of Part I, we get the following homogeneity property for the functional $\mathscr{L}^{(1)}$:

[1] Recall that the values of the various functions defined on the crack front are to be taken at the point s if their argument is not explicitly specified.

[2] Note however that $\mathscr{L}^{(1/2)}$ still depends on the curvatures $\mathbf{C}$ and Γ of the surface and the front of the initial crack because $\mathbf{K}^{(1/2)}$, when expressed in terms of the loading $\mathscr{T}$, depends on them through the vectors of initial SIFs $\mathbf{K}$ and non-singular stresses $\mathbf{T}$ [see eqn (19) of Part I].

$$\mathscr{L}^{(1)}[\lambda R, \mathbf{C}/\lambda, \Gamma/\lambda, \varphi, \varphi'/\lambda, a^*/\sqrt{\lambda}, C^*/\lambda, \eta'/(\lambda\eta); \mathscr{T}] = \frac{1}{\sqrt{\lambda}} \mathscr{L}^{(1)}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \eta'/\eta; \mathscr{T}]$$

for all positive λ, so that for $\lambda = 1/R$,

$$\mathscr{L}^{(1)}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \eta'/\eta; \mathscr{T}(R)] = \mathscr{L}^{(1)}[1, R\mathbf{C}, R\Gamma, \varphi, R\varphi', \sqrt{R}\, a^*, RC^*, R\eta'/\eta; \mathscr{T}(R)/\sqrt{R}]. \quad (5)$$

But, by eqn (2) of Part I, $\mathscr{T}(R)$ admits an expansion in powers of R for $R \to 0$ of the following type:

$$\mathscr{T}(R) = \frac{K_p}{\sqrt{R}}\{\mathbf{f}^p(\psi,\chi)\} + T_p\{\mathbf{g}^p(\psi,\chi)\} + [B_p\{\mathbf{h}^p(\psi,\chi)\} + K'_p\{\mathbf{l}^p(\psi,\chi)\} + C_{\lambda\mu}K_p\{\mathbf{m}^{p\lambda\mu}(\psi,\chi)\} + \Gamma K_p\{\mathbf{n}^p(\psi,\chi)\}]\sqrt{R} + O(R). \quad (6)$$

Expanding the right-hand side of eqn (5) in powers of R and using eqn (6), one gets

$$\begin{aligned}
&\mathscr{L}^{(1)}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \eta'/\eta; \mathscr{T}(R)] = \frac{1}{R}\cdot K_p \mathscr{L}^{(1)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi)\}] \\
&+ \frac{1}{\sqrt{R}}\cdot T_p \mathscr{L}^{(1)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{g}^p(\psi,\chi)\}] \\
&+ \frac{1}{\sqrt{R}}\cdot a^* K_p \frac{\partial \mathscr{L}^{(1)}}{\partial a^*}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi)\}] \\
&+ B_p \mathscr{L}^{(1)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{h}^p(\psi,\chi)\}] + K'_p \mathscr{L}^{(1)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{l}^p(\psi,\chi)\}] \\
&+ C_{\lambda\mu} K_p \frac{\partial \mathscr{L}^{(1)}}{\partial C_{\lambda\mu}}[1, \mathbf{C}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi) + C_{\nu\kappa}\mathbf{m}^{p\nu\kappa}(\psi,\chi)\}]_{\mathbf{C}=\mathbf{0}} \\
&+ \Gamma K_p \frac{\partial \mathscr{L}^{(1)}}{\partial \Gamma}[1, \mathbf{0}, \Gamma, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi) + \Gamma\mathbf{n}^p(\psi,\chi)\}]_{\Gamma=0} \\
&+ \varphi' K_p \frac{\partial \mathscr{L}^{(1)}}{\partial \varphi'}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi)\}] + a^* T_p \frac{\partial \mathscr{L}^{(1)}}{\partial a^*}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{g}^p(\psi,\chi)\}] \\
&+ C^* K_p \frac{\partial \mathscr{L}^{(1)}}{\partial C^*}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi)\}] \\
&+ \frac{\eta'}{\eta} K_p \frac{\partial \mathscr{L}^{(1)}}{\partial(\eta'/\eta)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi)\}] \\
&+ \frac{a^{*2}}{2} K_p \frac{\partial \mathscr{L}^{(1)}}{\partial a^{*2}}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi)\}] + O(\sqrt{R}). \quad (7)
\end{aligned}$$

Multiplying the right-hand side here by η and adding the term $\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)]$, one gets

$\mathbf{K}^{(1)}$ by eqn (4), and this quantity is by definition independent of R. Hence the divergent terms proportional to $R^{-1/2}$ and R^{-1} in this right-hand side must necessarily (once multiplied by η) cancel out with some corresponding terms in the expansion of $\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)]$ in powers of R. In fact, we noted at the beginning of Section 6 of Part I that as a consequence of the lack of dependence of the SIFs K_p^* just after the kink upon a^*, $\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}]$ is also independent of that parameter. Furthermore, eqn (13) of Part I implies that $\mathscr{T}^{(1)}(R)$ does not depend on it either. It follows that $\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)]$ is independent of a^*. Also, eqns (13) and (8) of Part I show that $\mathscr{T}^{(1)}(R)$ depends on the loading only through the initial SIFs and is not influenced by the non-singular stresses. It follows that the two terms proportional to $R^{-1/2}$ in the right-hand side of eqn (7), once multiplied by η, can have no equivalent in the expansion of $\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)]$ in powers of R and are therefore necessarily zero.

However, the same cannot be said of the term proportional to R^{-1}. Let us therefore define the *principal part*, noted PP, of $\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)]$, as

$$PP\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)] \equiv \lim_{R\to 0} \left\{ \mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)] + \frac{1}{R} \cdot K_p \eta \mathscr{L}^{(1)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi)\}] \right\} \quad (8)$$

(the existence of the limit is guaranteed by the very fact that the left-hand side of eqn (4) is independent of R). Equations (4) and (7) then yield, in the limit $R \to 0$:

$$\begin{aligned}
\mathbf{K}^{(1)} &= PP\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)] \\
&+ B_p \eta \mathscr{L}^{(1)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{h}^p(\psi, \chi)\}] + K'_p \eta \mathscr{L}^{(1)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{l}^p(\psi, \chi)\}] \\
&+ C_{\lambda\mu} K_p \eta \frac{\partial \mathscr{L}^{(1)}}{\partial C_{\lambda\mu}} [1, \mathbf{C}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi) + C_{\nu\kappa} \mathbf{m}^{p\nu\kappa}(\psi, \chi)\}]_{\mathbf{C}=\mathbf{0}} \\
&+ \Gamma K_p \eta \frac{\partial \mathscr{L}^{(1)}}{\partial \Gamma} [1, \mathbf{0}, \Gamma, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi) + \Gamma \mathbf{n}^p(\psi, \chi)\}]_{\Gamma=0} \\
&+ \varphi' K_p \eta \frac{\partial \mathscr{L}^{(1)}}{\partial \varphi'} [1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi)\}] \\
&+ a^* T_p \eta \frac{\partial \mathscr{L}^{(1)}}{\partial a^*} [1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{g}^p(\psi, \chi)\}] \\
&+ C^* K_p \eta \frac{\partial \mathscr{L}^{(1)}}{\partial C^*} [1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi)\}] \\
&+ K_p \eta' \frac{\partial \mathscr{L}^{(1)}}{\partial (\eta'/\eta)} [1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi)\}] \\
&+ \frac{a^{*2}}{2} K_p \eta \frac{\partial \mathscr{L}^{(1)}}{\partial a^{*2}} [1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi)\}]. \quad (9)
\end{aligned}$$

All terms here are in explicit form except for the first one. The problem is thus reduced to finding a more explicit formula for $PP\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{F}^{(1)}(R)]$.

3. Expression of $PP\mathscr{L}^*(\mathscr{F}^{(1)})$

The derivative, for $\varepsilon = 0$, of the displacement with respect to ε, is given by eqn (12) of Part I. We introduce, following Rice (1985), the diagonal matrix

$$\mathbf{\Lambda} \equiv (\Lambda_{pq}) \equiv \frac{1}{E}\begin{bmatrix} 1-v^2 & 0 & 0 \\ 0 & 1-v^2 & 0 \\ 0 & 0 & 1+v \end{bmatrix} \tag{10}$$

(where E and v are Young's modulus and Poisson's ratio), and the notation $\mathbf{K}_p^*(\Omega, s', M) \equiv [K_p^{(1)*}, K_p^{(2)*}, K_p^{(3)*}](\Omega, s', M)$. Remember that $K_p^{(i)*}(\Omega, s', M)$ here denotes the p-th SIF just after the kink at the point s' of the crack front arising from application of some unit point force in the direction $\mathbf{E}_i \equiv \partial\mathcal{O}\mathbf{M}/\partial X_i$ on the point M, $\partial\Omega_u$ and $\partial\Omega_t$ being simultaneously clamped and free of tractions. With this notation, one puts eqn (12) of Part I in vectorial form:

$$\frac{\partial \mathbf{u}}{\partial \varepsilon}(M, \varepsilon = 0) = \int_{\mathscr{F}} 2\Lambda_{qr} K_q^*(s') \mathbf{K}_r^*(\Omega, s', M) \eta(s')\, \mathrm{d}s'. \tag{11}$$

To obtain the corresponding traction field $\mathscr{F}^{(1)}$ on the boundary $\partial\mathscr{S}(s, R)$ of the sphere of centre s and radius R, one must evaluate the symmetrized gradient of this new "displacement", apply the elasticity operator to get the stresses and contract the result with the unit outward normal vector to the sphere. This succession of operations defines a first-order linear differential operator which will be denoted $\mathbf{L}_M$ (the index M underlines the dependence upon the position, which arises from that upon the local direction of the unit normal vector). Applying this operator to both sides of eqn (11), one gets the following expression of $\mathscr{F}^{(1)}$:

$$\mathscr{F}^{(1)}(R) \equiv \mathscr{F}^{(1)}(s, R) \equiv \int_{\mathscr{F}} 2\Lambda_{qr} K_q^*(s') \{\mathbf{L}_M \cdot \mathbf{K}_r^*(\Omega, s', M)\}_{\partial\mathscr{S}(s,R)} \eta(s')\, \mathrm{d}s' \tag{12}$$

where the notation $\{\cdots\}_{\partial\mathscr{S}(s,R)}$ indicates a traction field exerted on $\partial\mathscr{S}(s, R)$.

3.1. *Case of an identically zero kink angle*

This special case includes in particular that of a planar crack with a coplanar extension, which was studied by former authors (see the Introduction). It is considered first for simplicity; but, as will be seen, extending the reasoning to arbitrary kinked and curved cracks does not necessitate introducing new ideas, it only makes notations heavier.

By eqn (12), the components of the quantity $\mathscr{L}^*(\mathscr{F}^{(1)})$ of interest can be written as

$$\mathscr{L}_p^*[R, \mathbf{C}, \Gamma, \varphi = 0; \mathscr{T}^{(1)}(R)] \equiv \mathscr{L}_p[R, \mathbf{C}, \Gamma; \mathscr{T}^{(1)}(R)] = \int_{\mathscr{F}} Z_{pq}(\Omega, R, s, s') K_q(s') \eta(s') \, \mathrm{d}s' \quad (13)$$

where

$$Z_{pq}(\Omega, R, s, s') \equiv 2\Lambda_{qr} \mathscr{L}_p[R, \mathbf{C}, \Gamma; \{\mathbf{L}_M \cdot \mathbf{K}_r(\Omega, s', M)\}_{\partial\mathscr{S}(s,R)}]. \quad (14)$$

K_q^* and $\mathbf{K}_r^*$ here have simply become K_q and $\mathbf{K}_r$, since there is no kink. Similarly, the notation $\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi = 0; \mathscr{T}]$ has been replaced by the simpler and logical one $\mathscr{L}[R, \mathbf{C}, \Gamma; \mathscr{T}]$. Note that the operator $\mathbf{Z}$ defined by eqn (14) is independent of the curvature parameters $a^*(s')$ and $C^*(s')$ of the crack extension, since its definition involves only quantities relative to the situation prior to propagation of the crack.

Let us briefly sketch the difficulties to be encountered when taking the limit $R \to 0$ in the right-hand side of eqn (13). Even if the quantity $Z_{pq}(\Omega, R, s, s')$ is supposed to possess some limit $Z_{pq}(\Omega, s, s')$ for $R \to 0$, there is no reason to think that its convergence towards that limit will be uniform with respect to s' for s' close to s; it is in fact shown in Appendix B that it is *not* uniform. As a result, the limit of the integral of $Z_{pq}(\Omega, R, s, s')$ (times some other factors) over the crack front will not simply be the integral of $Z_{pq}(\Omega, s, s')$ (times the same factors). Also, the "kernel" $Z_{pq}(\Omega, s, s')$ will represent in some way the effect of the crack advance $\delta(s')$ at the point s' on the variation of the SIFs at the point s, and as such is bound to be singular (divergent) for $s' \to s$.

The treatment will rely on three lemmas. The proofs of these lemmas are quite technical, and for this reason relegated to Appendix A (for Lemma 1), Appendix B (for Lemma 2) and Appendices C and D (for Lemma 3). What these lemmas say is as follows.

Lemma 1. $D(s, s')$ denoting the Cartesian distance between the points s and s', the quantity $Z_{pq}(\Omega, R, s, s')$ is in reality independent of R for $R < D(s, s')$; its value in such conditions will be denoted $Z_{pq}(\Omega, s, s')$.[3] As a consequence, whatever the open interval $\mathscr{I}$ of $\mathscr{F}$ containing the point s, one has

$$\lim_{R \to 0} \int_{\mathscr{F} - \mathscr{I}} \mathbf{Z}(\Omega, R, s, s') \cdot \mathbf{K}(s') \eta(s') \, \mathrm{d}s' = \int_{\mathscr{F} - \mathscr{I}} \mathbf{Z}(\Omega, s, s') \cdot \mathbf{K}(s') \eta(s') \, \mathrm{d}s'. \quad (15)$$

Lemma 2. The function (of s') $\mathbf{Z}(\Omega, s, s')(s' - s)^2$ is continuous and finite at the point $s' = s$, and its value at that point is universal (it depends only on Poisson's ratio).

Lemma 3. If the propagation rate $\eta(s')$ is $O((s' - s)^2)$ in the vicinity of the point $s' = s$, then

$$\lim_{R \to 0} \int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s') \cdot \mathbf{K}(s') \eta(s') \, \mathrm{d}s' = \int_{\mathscr{F}} \mathbf{Z}(\Omega, s, s') \cdot \mathbf{K}(s') \eta(s') \, \mathrm{d}s' \quad (16)$$

(note that the integral in the right-hand side is convergent since $\mathbf{Z}(\Omega, s, s')$ is $O((s' - s)^{-2})$ by Lemma 2).

[3] This notation is consistent with the definition of $Z_{pq}(\Omega, s, s')$ as the limit of $Z_{pq}(\Omega, R, s, s')$ for $R \to 0$.

Combining eqns (8) and (13), one gets

$$PP\mathscr{L}[R,\mathbf{C},\Gamma;\mathscr{F}^{(1)}(R)] = \lim_{R\to 0}\Bigg[\int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s')\cdot\mathbf{K}(s')\eta(s')\,\mathrm{d}s' + \frac{1}{R}\cdot\eta(s)\mathscr{L}^{(1)}(1,\mathbf{0},0,\varphi(s)=0,0,0,0,0;K_p(s)\{\mathbf{f}^p(\psi,\chi)\})\Bigg] \tag{17}$$

where indications of dependence upon s have been restored in order to avoid any ambiguity. Let us first consider a *uniform* advance of the crack front: $\eta(s')\equiv\eta(s)$, $\forall s'$. Then

$$PP\mathscr{L}[R,\mathbf{C},\Gamma;\mathscr{F}^{(1)}(R)] = \mathbf{A}(s)\eta(s)$$

where

$$\mathbf{A}(s)\equiv\lim_{R\to 0}\Bigg[\int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s')\cdot\mathbf{K}(s')\,\mathrm{d}s' + \frac{1}{R}\cdot\mathscr{L}^{(1)}(1,\mathbf{0},0,\varphi(s)=0,0,0,0,0;K_p(s)\{\mathbf{f}^p(\psi,\chi)\})\Bigg]. \tag{18}$$

The existence of the limit here results from the fact that $PP\mathscr{L}[R,\mathbf{C},\Gamma;\mathscr{F}^{(1)}(R)]$ exists and is finite for all possible advances of the crack front, including a uniform one.

Let us now come back to the case of an arbitrary function $\eta(s')$, and rewrite the term between brackets in the right-hand side of eqn (17) as

$$\Bigg[\int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s')\cdot\mathbf{K}(s')\,\mathrm{d}s' + \frac{1}{R}\cdot\mathscr{L}^{(1)}(1,\mathbf{0},0,\varphi(s)=0,0,0,0,0;K_p(s)\{\mathbf{f}^p(\psi,\chi)\})\Bigg]\eta(s) + \int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s')\cdot\mathbf{K}(s')[\eta(s')-\eta(s)]\,\mathrm{d}s'.$$

When R goes to 0, the term $[\cdots]\eta(s)$ tends towards $\mathbf{A}(s)\eta(s)$ and the remaining integral has a limit, since the whole expression has a limit equal to $PP\mathscr{L}(\mathscr{F}^{(1)})$. Therefore,

$$PP\mathscr{L}[R,\mathbf{C},\Gamma;\mathscr{F}^{(1)}(R)] = \mathbf{A}(s)\eta(s) + \lim_{R\to 0}\int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s')\cdot\mathbf{K}(s')[\eta(s')-\eta(s)]\,\mathrm{d}s'. \tag{19}$$

In order to evaluate the integral in the right-hand side of eqn (19), let us split it in the following way, where $\mathscr{I}\equiv\,]s-\sigma, s+\sigma[$ (σ being an arbitrary positive number, fixed for the moment but intended to be shrunk to 0 *in fine*):

$$\int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s')\cdot\mathbf{K}(s')[\eta(s')-\eta(s)]\,\mathrm{d}s' = \int_{\mathscr{F}-\mathscr{I}} \mathbf{Z}(\Omega, R, s, s')\cdot\mathbf{K}(s')[\eta(s')-\eta(s)]\,\mathrm{d}s' + \int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s')\cdot\mathbf{K}(s')[\eta(s')-\eta(s)]\,\mathrm{d}s'$$

$$= \int_{\mathscr{F}-\mathscr{I}} \mathbf{Z}(\Omega, R, s, s') \cdot \mathbf{K}(s')[\eta(s') - \eta(s)]\, \mathrm{d}s'$$

$$+ \int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s') \cdot \mathbf{K}(s')[\eta(s') - \eta(s) - \eta'(s)(s'-s)]\, \mathrm{d}s'$$

$$+ \eta'(s) \int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s') \cdot \mathbf{K}(s')(s'-s)\, \mathrm{d}s'.$$

Now let us take the limit $R \to 0$ (σ being fixed) in the last expression. By Lemma 1, the first integral tends towards the integral of the same quantity but with $\mathbf{Z}(\Omega, s, s')$ instead of $\mathbf{Z}(\Omega, R, s, s')$. The same is true of the second integral by Lemma 3 (the function η being assumed to be of class $\mathscr{C}^\infty$). Finally the third integral has a limit since we have seen that the left-hand side possesses one. It follows that

$$\lim_{R\to 0} \int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s') \cdot \mathbf{K}(s')[\eta(s') - \eta(s)]\, \mathrm{d}s' = \int_{\mathscr{F}-\mathscr{I}} \mathbf{Z}(\Omega, s, s') \cdot \mathbf{K}(s')[\eta(s') - \eta(s)]\, \mathrm{d}s'$$

$$+ \int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, s, s') \cdot \mathbf{K}(s')[\eta(s') - \eta(s) - \eta'(s)(s'-s)]\, \mathrm{d}s'$$

$$+ \eta'(s) \cdot \lim_{R\to 0} \int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s') \cdot \mathbf{K}(s')(s'-s)]\, \mathrm{d}s'.$$

Now let us shrink σ to 0. The first integral in the right-hand side tends to the Cauchy principal value (noted PV) of the same integrand. (Note that by Lemma 2, this integrand is $O((s'-s)^{-1})$ for $s' \to s$, so that the integral does exist in principal value.) The integrand in the second integral is bounded by Lemma 2, so that the integral tends to 0 in the limit $\sigma \to 0$. Finally the third integral is forced to tend towards some limit since the left-hand side is independent of σ. We therefore get

$$\lim_{R\to 0} \int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s') \cdot \mathbf{K}(s')[\eta(s') - \eta(s)]\, \mathrm{d}s'$$

$$= PV \int_{\mathscr{F}} \mathbf{Z}(\Omega, s, s') \cdot \mathbf{K}(s')[\eta(s') - \eta(s)]\, \mathrm{d}s' + \mathbf{B}(s)\eta'(s)$$

where

$$\mathbf{B}(s) \equiv \lim_{\sigma\to 0} \lim_{R\to 0} \int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s') \cdot \mathbf{K}(s')(s'-s)\, \mathrm{d}s'. \tag{20}$$

Combining this last result with eqn (19), one obtains

$$PP\mathscr{L}[R, \mathbf{C}, \Gamma; \mathscr{T}^{(1)}(R)] = \mathbf{A}(s)\eta(s) + \mathbf{B}(s)\eta'(s) + PV \int_{\mathscr{F}} \mathbf{Z}(\Omega, s, s') \cdot \mathbf{K}(s')[\eta(s') - \eta(s)]\, \mathrm{d}s'. \tag{21}$$

Equation (21), which is the essential result of this section, shows that $PP\mathscr{L}(\mathscr{T}^{(1)})$ is not an arbitrary functional of the propagation rate $\eta(s')$: it is the sum of a term proportional to the local value of η at the point s, another one proportional to its local derivative $\eta'(s)$ along the crack front, and an integral in principal value of the "fluctuation" $\eta(s')-\eta(s)$ over the front.

Extra information can be given on the quantities $\mathbf{A}(s)$, $\mathbf{B}(s)$ and $\mathbf{Z}(\Omega, s, s')$. First, *the quantity* $\mathbf{A}(s)$ *does not admit any universal expression*. More precisely, it is impossible to express it in terms of the parameters characterizing the *local geometry* and the *initial local* stress expansion; it depends upon the whole geometry of the body and the initial crack considered and must therefore be evaluated in each particular case. This is obvious since it was already true in the two-dimensional case (see Sumi et al., 1983; Leblond, 1989; Leguillon, 1993). *The "kernel"* $\mathbf{Z}(\Omega, s, s')$ *is no more universal*, as symbolized by the presence of the argument "Ω" (although, as stated by Lemma 2, its asymptotic behavior for $s' \to s$ is universal). Indeed the components of $\mathbf{K}_r(\Omega, s', M)$ are SIFs at the point s' of the initial crack front generated by some unit point forces exerted on the point M, $\partial\Omega_u$ being clamped and $\partial\Omega_t$ free of tractions; therefore they depend upon the whole geometry of Ω, and the same is true of $\mathbf{Z}(\Omega, s, s')$ by eqn (14) and the definition of $\mathbf{Z}(\Omega, s, s')$ as the constant value of $\mathbf{Z}(\Omega, R, s, s')$ for $R < D(s, s')$.

In contrast, *the quantity* $\mathbf{B}(s)$ *does admit a universal expression in terms of the initial SIFs* (plus Poisson's ratio). To show this, let us split the right-hand side of eqn (20) as follows:

$$\mathbf{B}(s) = \left[\lim_{\sigma\to 0}\lim_{R\to 0}\int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s')(s'-s)\,\mathrm{d}s'\right]\cdot\mathbf{K}(s) + \lim_{\sigma\to 0}\lim_{R\to 0}\int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s')\cdot[\mathbf{K}(s')-\mathbf{K}(s)](s'-s)\,\mathrm{d}s'.$$

By Lemma 3, the limit of $\int_{s-\sigma}^{s+\sigma}\mathbf{Z}(\Omega, R, s, s')\cdot[\mathbf{K}(s')-\mathbf{K}(s)](s'-s)\,\mathrm{d}s'$ for $R\to 0$ is $\int_{s-\sigma}^{s+\sigma}\mathbf{Z}(\Omega, s, s')\cdot[\mathbf{K}(s')-\mathbf{K}(s)](s'-s)\,\mathrm{d}s'$. By Lemma 2, the integrand here is bounded so that the limit of the integral for $\sigma\to 0$ is zero. Therefore the preceding expression of $\mathbf{B}(s)$ becomes

$$\mathbf{B}(s) = \left[\lim_{\sigma\to 0}\lim_{R\to 0}\int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s')(s'-s)\,\mathrm{d}s'\right]\cdot\mathbf{K}(s). \tag{20'}$$

It is proved in Appendix E that the term $[\cdots]$ here depends only on Poisson's ratio, whence follows the universal character of $\mathbf{B}(s)$.

3.2. *Case of an arbitrary kinked crack extension*

We now come back to the general case where $\varphi(s') \not\equiv 0$, the treatment of which will require only moderate additional effort. Equation (8) of Part I yields $K^*_m(s') = F_{mq}(\varphi(s'))K_q(s')$ and $K^{(i)}_n{}^*(\Omega, s', M) = F_{nr}(\varphi(s'))K^{(i)}_r(\Omega, s', M) \Rightarrow \mathbf{K}^*_n(\Omega, s', M) = F_{nr}(\varphi(s'))\ \mathbf{K}_r(\Omega, s', M)$. It follows from there and eqn (12) that

$$\mathscr{T}^{(1)}(R) \equiv \mathscr{T}^{(1)}(s, R) = \int_{\mathscr{F}} 2\Lambda_{mn}K^*_m(s')\{\mathbf{L}_M\cdot\mathbf{K}^*_n(\Omega, s', M)\}_{\partial\mathscr{S}(s,R)}\eta(s')\,\mathrm{d}s'$$
$$= \int_{\mathscr{F}} 2\Lambda_{mn}F_{mq}(\varphi(s'))K_q(s')F_{nr}(\varphi(s'))\{\mathbf{L}_M\cdot\mathbf{K}_r(\Omega, s', M)\}_{\partial\mathscr{S}(s,R)}\eta(s')\,\mathrm{d}s$$

so that

$$\mathscr{L}^*_p[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)] = \int_{\mathscr{F}} 2\Lambda_{mn} F_{mq}(\varphi(s')) F_{nr}(\varphi(s')) \mathscr{L}^*_p[R, \mathbf{C}, \Gamma, \varphi; \{\mathbf{L}_M \cdot \mathbf{K}_r(\Omega, s', M)\}_{\partial\mathscr{S}(s,R)}] K_q(s')\eta(s')\,\mathrm{d}s'.$$

But eqn (8) of Part I also implies that

$$\mathscr{L}^*_p[R, \mathbf{C}, \Gamma, \varphi; \{\mathbf{L}_M \cdot \mathbf{K}_r(\Omega, s', M)\}_{\partial\mathscr{S}(s,R)}] = F_{ps}(\varphi(s)) \mathscr{L}_s[R, \mathbf{C}, \Gamma; \{\mathbf{L}_M \cdot \mathbf{K}_r(\Omega, s', M)\}_{\partial\mathscr{S}(s,R)}]$$

where, as in Section 3.1 above, the notation $\mathscr{L}^*(R, \mathbf{C}, \Gamma, \varphi = 0; \mathscr{T})$ has been replaced by $\mathscr{L}(R, \mathbf{C}, \Gamma; \mathscr{T})$. It follows that

$$\mathscr{L}^*_p[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)] = \int_{\mathscr{F}} Z_{pq}(\Omega, R, s, s', \varphi(s), \varphi(s')) K_q(s')\eta(s')\,\mathrm{d}s' \tag{22}$$

where

$$Z_{pq}(\Omega, R, s, s', \varphi(s), \varphi(s')) = 2F_{ps}(\varphi(s)) \mathscr{L}_s[R, \mathbf{C}, \Gamma; \{\mathbf{L}_M \cdot \mathbf{K}_r(\Omega, s', M)\}_{\partial\mathscr{S}(s,R)}] F_{nr}(\varphi(s')) \Lambda_{mn} F_{mq}(\varphi(s')). \tag{23}$$

Note that the very definition (23) of the operator $\mathbf{Z}$ implies that it is independent of the curvature parameters $a^*(s')$, $C^*(s')$ of the crack extension, just as in the case of an identically zero kink angle (see Section 3.1).

The operator $\mathbf{Z}(\Omega, R, s, s', \varphi(s), \varphi(s'))$ happens to be expressible in terms of the kink angles $\varphi(s)$, $\varphi(s')$ and the same operator but for zero angles $\varphi(s)$, $\varphi(s')$. Indeed, for $\varphi(s) = \varphi(s') = 0 \Rightarrow \mathbf{F}(\varphi(s)) = \mathbf{F}(\varphi(s')) = \mathbf{1}$,[4] eqn (23) yields

$$Z_{pq}(\Omega, R, s, s', 0, 0) \equiv Z_{pq}(\Omega, R, s, s') = 2\mathscr{L}_p[R, \mathbf{C}, \Gamma; \{\mathbf{L}_M \cdot \mathbf{K}_n(\Omega, s', M)\}_{\partial\mathscr{S}(s,R)}] \Lambda_{nq}$$

where the symmetry of the matrix $\mathbf{\Lambda}$ has been used. Inversion of this equation yields

$$2\mathscr{L}_s[R, \mathbf{C}, \Gamma; \{\mathbf{L}_M \cdot \mathbf{K}_r(\Omega, s', M)\}_{\partial\mathscr{S}(s,R)}] = Z_{st}(\Omega, R, s, s') \Lambda^{-1}_{tr};$$

inserting this result into eqn (23), one gets

$$Z_{pq}(\Omega, R, s, s', \varphi(s), \varphi(s')) = F_{ps}(\varphi(s)) Z_{st}(\Omega, R, s, s') \Lambda^{-1}_{tr} F_{nr}(\varphi(s')) \Lambda_{nm} F_{mq}(\varphi(s'))$$
$$\Rightarrow \mathbf{Z}(\Omega, R, s, s', \varphi(s), \varphi(s')) = \mathbf{F}(\varphi(s)) \cdot \mathbf{Z}(\Omega, R, s, s') \cdot \mathbf{\Lambda}^{-1} \cdot \mathbf{F}^T(\varphi(s')) \cdot \mathbf{\Lambda} \cdot \mathbf{F}(\varphi(s'))$$

where $\mathbf{F}^T$ denotes the transpose of $\mathbf{F}$. But, because of the form of the matrices $\mathbf{F}$ and $\mathbf{\Lambda}$ [see eqns (9) of Part I and (10) of the present paper], $\mathbf{F}^T$ and $\mathbf{\Lambda}$ happen to commute so that the preceding equation takes the simpler form

$$\mathbf{Z}(\Omega, R, s, s', \varphi(s), \varphi(s')) = \mathbf{F}(\varphi(s)) \cdot \mathbf{Z}(\Omega, R, s, s') \cdot (\mathbf{F}^T \cdot \mathbf{F})(\varphi(s')) \tag{24}$$

(note that the argument of the first "$\mathbf{F}$" is different from that of the other two).

[4] The property $\mathbf{F}(0) = \mathbf{1}$ is obvious: it just means that the SIFs just before and just after the kink are identical if there is no kink.

The rest of the treatment is based on eqns (22) and (24) plus three new lemmas.

Lemma 1′. The operator $\mathbf{Z}(\Omega, R, s, s', \varphi(s), \varphi(s'))$ is in fact independent of R for $R < D(s, s')$, its value being then noted $\mathbf{Z}(\Omega, s, s', \varphi(s), \varphi(s'))$. As a consequence, for every open interval $\mathscr{I}$ of the crack front containing the point s, one has

$$\lim_{R\to 0}\int_{\mathscr{F}-\mathscr{I}} \mathbf{Z}(\Omega, R, s, s', \varphi(s), \varphi(s'))\cdot\mathbf{K}(s')\eta(s')\,\mathrm{d}s' = \int_{\mathscr{F}-\mathscr{I}} \mathbf{Z}(\Omega, s, s', \varphi(s), \varphi(s'))\cdot\mathbf{K}(s')\eta(s')\,\mathrm{d}s'.$$

This lemma is a trivial consequence of Lemma 1 and eqn (24). This equation also implies that

$$\mathbf{Z}(\Omega, s, s', \varphi(s), \varphi(s')) = \mathbf{F}(\varphi(s))\cdot\mathbf{Z}(\Omega, s, s')\cdot(\mathbf{F}^T\cdot\mathbf{F})(\varphi(s')). \tag{25}$$

This result shows that *the operator* $\mathbf{Z}(\Omega, s, s', \varphi(s), \varphi(s'))$ *depends upon the kink angles* $\varphi(s)$, $\varphi(s')$ *in a universal way, through the same operator* $\mathbf{F}$ *that connects the SIFs just before and just after the kink*. In other words, the non-universal character of this operator is "concentrated" in its value for zero kink angles $\varphi(s)$, $\varphi(s')$.

Lemma 2′. The function (of s') $\mathbf{Z}(\Omega, s, s', \varphi(s), \varphi(s'))(s'-s)^2$ is continuous and finite at the point $s' = s$, and its value at that point is universal (it depends only on $\varphi(s)$ and Poisson's ratio).

Again, this is a trivial consequence of Lemma 2 and eqn (25).

Lemma 3′. If the propagation rate $\eta(s')$ is $O((s'-s)^2)$ for $s' \to s$, then

$$\lim_{R\to 0}\int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s', \varphi(s), \varphi(s'))\cdot\mathbf{K}(s')\eta(s')\,\mathrm{d}s' = \int_{\mathscr{F}} \mathbf{Z}(\Omega, s, s', \varphi(s), \varphi(s'))\cdot\mathbf{K}(s')\eta(s')\,\mathrm{d}s'.$$

This is a consequence of Lemma 3 plus the fact that each component of the integrand in the left-hand side can be decomposed into products of some component of the operator $\mathbf{Z}(\Omega, R, s, s')$ times some term which is $O((s'-s)^2)$.

Using these three lemmas and following the same reasoning as in the case where $\varphi(s') \equiv 0$ (see Section 3.1), one finds that

$$\begin{aligned} PP\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}^{(1)}(R)] &= \mathbf{A}(s)\eta(s) + \mathbf{B}(s)\eta'(s) \\ &\quad + PV\int_{\mathscr{F}} \mathbf{Z}(\Omega, s, s', \varphi(s), \varphi(s'))\cdot\mathbf{K}(s')[\eta(s')-\eta(s)]\,\mathrm{d}s' \end{aligned} \tag{26}$$

where

$$\begin{aligned} \mathbf{A}(s) \equiv \lim_{R\to 0}\Bigg[&\int_{\mathscr{F}} \mathbf{Z}(\Omega, R, s, s', \varphi(s), \varphi(s'))\cdot\mathbf{K}(s')\,\mathrm{d}s' \\ &+ \frac{1}{R}\cdot\mathscr{L}^{(1)}(1, \mathbf{0}, 0, \varphi(s), 0, 0, 0, 0; K_p(s)\{\mathbf{f}^p(\psi, \chi)\}\Bigg] \end{aligned} \tag{27}$$

and

$$\mathbf{B}(s) \equiv \lim_{\sigma\to 0}\lim_{R\to 0}\int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s', \varphi(s), \varphi(s'))\cdot\mathbf{K}(s')(s'-s)\,\mathrm{d}s'. \tag{28}$$

Equation (26) stands as the extension of the basic result (21) applicable to the special case of an identically zero kink angle, to the general case of an arbitrarily kinked crack extension.

Just as in the case where $\varphi(s') \equiv 0$, the quantity $\mathbf{B}(s)$ admits a universal expression. To show this, let us first remark that exactly as we did before, we may replace $\mathbf{K}(s')$ by $\mathbf{K}(s)$ in the integral of eqn (28). Writing then $\mathbf{Z}(\Omega, R, s, s', \varphi(s), \varphi(s'))(s'-s)$, using eqn (24), as

$$\mathbf{F}(\varphi(s))\cdot\mathbf{Z}(\Omega, R, s, s')\cdot(\mathbf{F}^T\cdot\mathbf{F})(\varphi(s))(s'-s)$$
$$+\mathbf{F}(\varphi(s))\cdot\mathbf{Z}(\Omega, R, s, s')\cdot[(\mathbf{F}^T\cdot\mathbf{F})(\varphi(s'))-(\mathbf{F}^T\cdot\mathbf{F})(\varphi(s))](s'-s)$$

and noting that again, the integral of the last term vanishes in the double limit $R\to 0$, then $\sigma\to 0$, one sees that the expression (28) of $\mathbf{B}(s)$ becomes

$$\mathbf{B}(s) = \mathbf{F}(\varphi(s))\cdot\left[\lim_{\sigma\to 0}\lim_{R\to 0}\int_{s-\sigma}^{s+\sigma}\mathbf{Z}(\Omega, R, s, s')(s'-s)\,\mathrm{d}s'\right]\cdot(\mathbf{F}^T\cdot\mathbf{F})(\varphi(s))\cdot\mathbf{K}(s). \tag{28'}$$

This formula, which extends eqn (20′) to the general case where $\varphi(s') \not\equiv 0$, establishes the universal character of $\mathbf{B}(s)$, since the term $[\cdots]$ depends only on Poisson's ratio (see Appendix E).

4. First formula for $\mathbf{K}^{(1)}$—The operator $\mathbf{N}(\varphi)$

We shall directly envisage here the general case where $\varphi(s') \not\equiv 0$; the situation where $\varphi(s') \not\equiv 0$ will be treated as a special case.

Using eqns (9) and (26) and changing the order of some terms, one gets

$$\mathbf{K}^{(1)}(s) = \mathbf{A}(s)\eta(s) + B_p(s)\eta(s)\mathscr{L}^{(1)}[1, \mathbf{0}, 0, \varphi(s), 0, 0, 0, 0; \{\mathbf{h}^p(\psi,\chi)\}]$$
$$+K'_p(s)\eta(s)\mathscr{L}^{(1)}[1, \mathbf{0}, 0, \varphi(s), 0, 0, 0, 0; \{\mathbf{l}^p(\psi,\chi)\}]$$
$$+C_{\lambda\mu}(s)K_p(s)\eta(s)\frac{\partial\mathscr{L}^{(1)}}{\partial C_{\lambda\mu}}[1, \mathbf{C}, 0, \varphi(s), 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi)+C_{\nu\kappa}\mathbf{m}^{p\nu\kappa}(\psi,\chi)\}]_{\mathbf{C}=\mathbf{0}}$$
$$+\Gamma(s)K_p(s)\eta(s)\frac{\partial\mathscr{L}^{(1)}}{\partial\Gamma}[1, \mathbf{0}, \Gamma, \varphi(s), 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi)+\Gamma\mathbf{n}^p(\psi,\chi)\}]_{\Gamma=\mathbf{0}}$$
$$+\varphi'(s)K_p(s)\eta(s)\frac{\partial\mathscr{L}^{(1)}}{\partial\varphi'}[1, \mathbf{0}, 0, \varphi(s), 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi)\}]$$
$$+a^*(s)T_p(s)\eta(s)\frac{\partial\mathscr{L}^{(1)}}{\partial a^*}[1, \mathbf{0}, 0, \varphi(s), 0, 0, 0, 0; \{\mathbf{g}^p(\psi,\chi)\}]$$
$$+\frac{a^{*2}(s)}{2}K_p(s)\eta(s)\frac{\partial^2\mathscr{L}^{(1)}}{\partial a^{*2}}[1, \mathbf{0}, 0, \varphi(s), 0, 0, 0, 0; \{\mathbf{f}^p(\psi,\chi)\}]$$

$$+C^*(s)K_p(s)\eta(s)\frac{\partial\mathscr{L}^{(1)}}{\partial C^*}[1,\mathbf{0},0,\varphi(s),0,0,0,0;\{\mathbf{f}^p(\psi,\chi)\}]$$

$$+\mathbf{B}(s)\eta'(s)+K_p(s)\eta'(s)\frac{\partial\mathscr{L}^{(1)}}{\partial(\eta'/\eta)}[1,\mathbf{0},0,\varphi(s),0,0,0,0;\{\mathbf{f}^p(\psi,\chi)\}]$$

$$+PV\int_{\mathscr{F}}\mathbf{Z}(\Omega,s,s',\varphi(s),\varphi(s'))\cdot\mathbf{K}(s')[\eta(s')-\eta(s)]\,\mathrm{d}s' \qquad (29)$$

where the argument s has systematically been re-introduced in order to avoid any ambiguity. In the right-hand side here, three groups of terms can be distinguished according to the way they are influenced by the propagation rate. The first one, which includes terms 1–9, involves only $\eta(s)$. The second group consists of terms 10 and 11 and is proportional to $\eta'(s)$. Finally the third group consists only of the last, integral term involving the fluctuation $\eta(s')-\eta(s)$. A further distinction can be made among the terms belonging to the first group. Indeed, we shall see below that when one will want to use eqn (29), combined with some propagation criterion, to derive the values of the geometric parameters $\varphi(s)$, $a^*(s)$, $C^*(s)$, $\eta(s)$ of the crack extension along the crack front, $\varphi(s)$ and $a^*(s)$ will in fact already be known. One can therefore distinguish between the eight first terms which do not involve the unknown $C^*(s)$, and the ninth which does.

Let us first consider the case where the propagation rate is uniform: $\eta(s')\equiv\eta(s)\equiv\eta$, $\forall s'$, and $C^*(s)$ is zero. We then have, according to eqn (3) and since $\varepsilon\eta=\delta$:

$$\mathbf{K}(s,\varepsilon)=\mathbf{K}^*(s)+\mathbf{K}^{(1/2)}(s)\sqrt{\varepsilon\eta}+\mathbf{K}^{(1)}(s)\varepsilon+O(\varepsilon^{3/2})=\mathbf{K}^*(s)+\mathbf{K}^{(1/2)}(s)\sqrt{\delta}+\mathbf{K}^{(1)}(s)\frac{\delta}{\eta}+O(\varepsilon^{3/2})$$

where the expression (29) of $\mathbf{K}^{(1)}(s)$ reduces to its eight first terms. But since the crack advance is then characterized by the *single* parameter δ instead of a *function* $\delta(s')$, one may also write, with obvious notations:

$$\mathbf{K}(s,\delta)\equiv\mathbf{K}^*(s)+\left[\frac{\partial\mathbf{K}(s,\delta)}{\partial\sqrt{\delta}}\right]_{\delta(s')\equiv\delta(s)}\sqrt{\delta}+\frac{1}{2}\left[\frac{\partial^2\mathbf{K}(s,\delta)}{\partial(\sqrt{\delta})^2}\right]^{C^*(s)=0}_{\delta(s')\equiv\delta(s)}\delta+O(\delta^{3/2})$$

and comparison with the preceding formula shows that the eight first terms of the expression (29) of $\mathbf{K}^{(1)}$ can be identified with $\frac{1}{2}[\partial^2\mathbf{K}(s,\delta)/\partial(\sqrt{\delta})^2]^{C^*(s)=0}_{\delta(s')\equiv\delta(s)}\,\eta$.

Let us now come back to the general case, i.e. let us introduce a non-uniform propagation rate $\eta(s')$ and a non-zero $C^*(s')$. The eight first terms of the expression (29), remain unchanged,[5] so that gathering the terms of the second group and accounting for the fact that $\mathbf{B}(s)$ admits a universal expression in terms of the initial SIFs and the local kink angle [see eqn (28′)], one may write $\mathbf{K}^{(1)}(s)$ in the form

[5] For the term $\mathbf{A}(s)\eta(s)$, where $\mathbf{A}(s)$ is given by eqn (27), this is because, as mentioned in Section 3.2 above, the operator $\mathbf{Z}$ is independent of $C^*(s)$.

$$\mathbf{K}^{(1)}(s) \equiv \frac{1}{2}\left[\frac{\partial^2 \mathbf{K}(s,\delta)}{\partial(\sqrt{\delta})^2}\right]_{\delta(s')\equiv\delta(s)}^{C^*(s)=0} \eta(s) + C^*(s)\mathbf{M}(\varphi(s))\cdot\mathbf{K}(s)\eta(s) + \mathbf{N}(\varphi(s))\cdot\mathbf{K}(s)\eta'(s)$$

$$+ VP\int_{\mathscr{F}} \mathbf{Z}(\Omega, s, s', \varphi(s), \varphi(s'))\cdot\mathbf{K}(s')[\eta(s')-\eta(s)]\,\mathrm{d}s' \quad (30)$$

where $\mathbf{M}(\varphi)$ and $\mathbf{N}(\varphi)$ are two new universal operators.

Equation (30) is the first fundamental formula for $\mathbf{K}^{(1)}$ we wished to establish. Note that it comprises terms of both *local* and *non-local* character. The unknown function $\eta(s')$ appears in the local terms through its value and that of its first derivative at the point s, and in the non-local one through the fluctuation $\eta(s')-\eta(s)$. On the other hand, the unknown function $C^*(s')$ appears only in a local term.

The first and fourth terms of the right-hand side of eqn (30) are of *non-universal* character[6] and will have to be evaluated in each particular case, but the second and third are universal. The operator $\mathbf{M}(\varphi)$ describes the influence of the curvature of the crack extension in the Ox_1x_2 plane and as such, can be evaluated by considering the two-dimensional problem of a straight initial crack with a kinked and curved extension in an infinite medium (just as for the calculation of the operator $\mathbf{H}(\varphi)$ describing the influence of the parameter a^* upon $\mathbf{K}^{(1/2)}$: see Part I, Section 6). This was done by Amestoy and Leblond (1992) for a plane loading, thus providing the (numerical) values of the components $M_{\mathrm{I,I}}(\varphi)$, $M_{\mathrm{I,II}}(\varphi)$, $M_{\mathrm{II,I}}(\varphi)$, $M_{\mathrm{II,II}}(\varphi)$. Leblond also considered the anti-plane case in an unpublished work, and obtained an analytical formula for $M_{\mathrm{III,III}}(\varphi)$. These numerical values and analytical formula are given in Appendix F. The remaining components $M_{\mathrm{I,III}}(\varphi)$, $M_{\mathrm{II,III}}(\varphi)$, $M_{\mathrm{III,I}}(\varphi)$, $M_{\mathrm{III,II}}(\varphi)$ are easily shown to be zero by considering a symmetry with respect to the Ox_1x_2 plane.

In contrast with the operator $\mathbf{M}(\varphi)$, the operator $\mathbf{N}(\varphi)$ cannot be deduced from consideration of plane problems, since it describes the influence of the derivative of the propagation rate along the crack front. The values of its components were unknown until very recently, except in the special case of a zero kink angle. Indeed in that case they can be deduced from the works of Rice (1985), Gao and Rice (1986, 1987a, b) and Gao (1988) concerning coplanar extensions of initially planar cracks; these values read as follows:

$$N_{\mathrm{II,III}}(0) = -\frac{2}{2-\nu}, \quad N_{\mathrm{III,II}}(0) = \frac{2(1-\nu)}{2-\nu}, \quad (31)$$

the other components being zero. The full calculation of that operator, for arbitrary values of the kink angle, was carried out in Lazarus' (1997) thesis, using a method first proposed by Mouchrif (1994) in his own thesis (see Section 5 below), and her numerical results are presented in Appendix F. [They agree with eqns (31) for $\varphi = 0$.]

Also, Appendix F provides the non-zero components of the non-universal operator $\mathbf{Z}(\Omega, s, s')$

[6] Remember that this means that they cannot be expressed solely in terms of the local geometric parameters and the coefficients of the local stress expansion before the kink, but depend on the whole geometry of the body and the initial crack considered.

taken from the work of Gao and Rice (1986), for the important special case of a semi-infinite plane crack in an infinite body.

In the special case of a *regular propagation* ($\varphi(s') \equiv 0$, $a^*(s') \equiv 0$[7]), $\mathbf{K}^{(1/2)}$ is zero by virtue of eqn (19) of Part I plus the property $\mathbf{G}(0) = \mathbf{0}$ (see Amestoy and Leblond, 1992). Therefore, if in addition the propagation rate is uniform and $C^*(s) = 0$, the expansion of $\mathbf{K}(s, \delta)$ takes the simple form, with obvious notations:

$$\mathbf{K}(s,\delta) \equiv \mathbf{K}(s) + \left[\frac{\partial \mathbf{K}(s,\delta)}{\partial \delta}\right]_{\delta(s')\equiv\delta(s)}^{C^*(s)=0} \delta + O(\delta^{3/2})$$

so that it becomes logical, and also simpler, to change the notation $\frac{1}{2}[\partial^2\mathbf{K}(s,\delta)/\partial(\sqrt{\delta})^2]_{\delta(s')\equiv\delta(s)}^{C^*(s)=0}$ into $[\partial\mathbf{K}(s,\delta)/\partial\delta]_{\delta(s')\equiv\delta(s)}^{C^*(s)=0}$. For (a regular propagation but) arbitrary values of the propagation rate and the curvature parameter $C^*(s)$, eqn (30) then becomes

$$\mathbf{K}^{(1)}(s) \equiv \left[\frac{\partial \mathbf{K}(s,\delta)}{\partial \delta}\right]_{\delta(s')\equiv\delta(s)}^{C^*(s)=0} \eta(s) + C^*(s)\mathbf{M}(0)\cdot\mathbf{K}(s)\eta(s)$$
$$+\mathbf{N}(0)\cdot\mathbf{K}(s)\eta'(s) + VP\int_{\mathscr{F}} \mathbf{Z}(\Omega, s, s')\cdot\mathbf{K}(s')[\eta(s')-\eta(s)]\,\mathrm{d}s'. \quad (30')$$

5. Second formula for $\mathbf{K}^{(1)}$—The operator $\mathbf{P}(\varphi)$

Formulae (29) and (30) for $\mathbf{K}^{(1)}$ say nothing about how the non-universal quantities $\mathbf{A}(s)$ and $\frac{1}{2}[\partial^2\mathbf{K}(s,\delta)/\partial(\sqrt{\delta})^2]_{\delta(s')\equiv\delta(s)}^{C^*(s)=0}$ are influenced by the function $\varphi(s')$ characterizing the direction of propagation at all points of the crack front. In general, this will not be a drawback because, as already mentioned and explained in Section 7 below, when one will make use of eqn (30) to derive the values of $C^*(s)$ and $\delta(s)$ along the crack front, the kink angle will already be known everywhere so that a numerical calculation of $\mathbf{A}(s)$ and $\frac{1}{2}[\partial^2\mathbf{K}(s,\delta)/\partial(\sqrt{\delta})^2]_{\delta(s')\equiv\delta(s)}^{C^*(s)=0}$ will always be possible. However, in a future theoretical study of crack front segmentation in mixed mode I+III, the kink angle will *not* be known *a priori* so that it will become necessary to know how these quantities depend upon the function $\varphi(s')$. The aim of this section is precisely to make this influence explicit.

By eqns (24) and (27), the value of $\mathbf{A}(s)$ for an arbitrary function $\varphi(s')$ can be related to that for a uniform function: $\varphi(s') \equiv \varphi(s)$, $\forall s'$, plus some corrective term as follows:

$$\mathbf{A}(s) = [\mathbf{A}(s)]_{\varphi(s')\equiv\varphi(s)} + \lim_{R\to 0}\int_{\mathscr{F}} \mathbf{F}(\varphi(s))\cdot\mathbf{Z}(\Omega, R, s, s')$$
$$\cdot[(\mathbf{F}^T\cdot\mathbf{F})(\varphi(s')) - (\mathbf{F}^T\cdot\mathbf{F})(\varphi(s))]\cdot\mathbf{K}(s')\,\mathrm{d}s.$$

To evaluate that limit, let us use the same kind of reasoning as above for the calculation of $PP\mathscr{L}^*(\mathscr{T}^{(1)})$, and decompose the crack front $\mathscr{F}$ into $\mathscr{F}-\mathscr{I}$ and $\mathscr{I} \equiv]s-\sigma, s+\sigma[$ where σ is an

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[7] Because a non-zero $a^*(s)$ would imply an infinite curvature of the crack extension at the point s of the front.

arbitrary positive number. In the integral over $\mathscr{F}-\mathscr{I}$, one can replace $\mathbf{Z}(\Omega, R, s, s')$ by $\mathbf{Z}(\Omega, s, s')$ in the limit $R \to 0$ (Lemma 1) and the resulting integral tends towards

$$PV\int_{\mathscr{F}} \mathbf{F}(\varphi(s)) \cdot \mathbf{Z}(\Omega, s, s') \cdot [(\mathbf{F}^T \cdot \mathbf{F})(\varphi(s')) - (\mathbf{F}^T \cdot \mathbf{F})(\varphi(s))] \cdot \mathbf{K}(s')\,\mathrm{d}s'$$

for $\sigma \to 0$ (the convergence of the integral in principal value being guaranteed by Lemma 2). The integral over $\mathscr{I}$ can be written in the form

$$\int_{s-\sigma}^{s+\sigma} \mathbf{F}(\varphi(s)) \cdot \mathbf{Z}(\Omega, R, s, s') \cdot \Bigg[(\mathbf{F}^T \cdot \mathbf{F})(\varphi(s')) - (\mathbf{F}^T \cdot \mathbf{F})(\varphi(s)) - \frac{\mathrm{d}}{\mathrm{d}s}[(\mathbf{F}^T \cdot \mathbf{F})(\varphi(s))](s'-s) \Bigg] \cdot \mathbf{K}(s')\,\mathrm{d}s' + \int_{s-\sigma}^{s+\sigma} \mathbf{F}(\varphi(s)) \cdot \mathbf{Z}(\Omega, R, s, s') \cdot \frac{\mathrm{d}}{\mathrm{d}s}[(\mathbf{F}^T \cdot \mathbf{F})(\varphi(s))] \cdot \mathbf{K}(s')(s'-s)\,\mathrm{d}s'.$$

The first integral here tends to 0 in the limit $R \to 0$, then $\sigma \to 0$ (Lemmas 2 and 3). In the second one, one may replace $\mathbf{K}(s')$ by $\mathbf{K}(s)$, again by Lemmas 2 and 3, and the resulting integral tends towards

$$\mathbf{F}(\varphi(s)) \cdot \left[\lim_{\sigma\to 0} \lim_{R\to 0} \int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s')(s'-s)\,\mathrm{d}s' \right] \cdot \frac{\mathrm{d}(\mathbf{F}^T \cdot \mathbf{F})}{\mathrm{d}\varphi}(\varphi(s)) \cdot \mathbf{K}(s)\varphi'(s)$$

in the limit $R \to 0$, then $\sigma \to 0$. Gathering these results, one sees that

$$\mathbf{A}(s) = [\mathbf{A}(s)]_{\varphi(s')\equiv\varphi(s)} + \mathbf{F}(\varphi(s)) \cdot PV\int_{\mathscr{F}} \mathbf{Z}(\Omega, s, s') \cdot [(\mathbf{F}^T \cdot \mathbf{F})(\varphi(s')) - (\mathbf{F}^T \cdot \mathbf{F})(\varphi(s))] \cdot \mathbf{K}(s')\,\mathrm{d}s' + \mathbf{F}(\varphi(s)) \cdot \left[\lim_{\sigma\to 0} \lim_{R\to 0} \int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s')(s'-s)\,\mathrm{d}s' \right] \cdot \frac{\mathrm{d}(\mathbf{F}^T \cdot \mathbf{F})}{\mathrm{d}\varphi}(\varphi(s)) \cdot \mathbf{K}(s)\varphi'(s).$$

Inserting this formula into eqn (29) and reordering and grouping some terms together, one gets:

$$\mathbf{K}^{(1)}(s) \equiv \frac{1}{2}\left[\frac{\partial^2 \mathbf{K}(s,\delta)}{\partial(\sqrt{\delta})^2}\right]^{C^*(s)=0}_{\delta(s')\equiv\delta(s),\varphi(s')\equiv\varphi(s)} \eta(s) + C^*(s)\mathbf{M}(\varphi(s)) \cdot \mathbf{K}(s)\eta(s) + \mathbf{N}(\varphi(s)) \cdot \mathbf{K}(s)\eta'(s) + \mathbf{P}(\varphi(s)) \cdot \mathbf{K}(s)\varphi'(s)\eta(s) + \mathbf{F}(\varphi(s)) \cdot PV\int_{\mathscr{F}} \mathbf{Z}(\Omega, s, s') \cdot [(\mathbf{F}^T \cdot \mathbf{F})(\varphi(s'))\eta(s') - (\mathbf{F}^T \cdot \mathbf{F})(\varphi(s))\eta(s)] \cdot \mathbf{K}(s')\,\mathrm{d}s'. \tag{32}$$

In this expression, the notation $\frac{1}{2}[\partial^2\mathbf{K}(s,\delta)/\partial(\sqrt{\delta})^2]^{C^*(s)=0}_{\delta(s')\equiv\delta(s),\varphi(s')\equiv\varphi(s)}$ represents the value of $\frac{1}{2}[\partial^2\mathbf{K}(s,\delta)/\partial(\sqrt{\delta})^2]^{C^*(s)=0}_{\delta(s')\equiv\delta(s)}$ for *a uniform kink angle* equal to $\varphi(s)$ and $\mathbf{P}(\varphi)$ is a new universal operator.

Equation (32) is the second fundamental formula for $\mathbf{K}^{(1)}(s)$ that we wished to establish. It splits the influence of the function $\varphi(s')$ into two kinds of terms, just as eqn (30) did for the function $\eta(s')$: *local ones* which involve only its value and that of its first derivative at the point s, and a *non-local* one proportional to an integral in principal value of the function $(\mathbf{F}^T\cdot\mathbf{F})(\varphi(s'))\eta(s')-(\mathbf{F}^T\cdot\mathbf{F})(\varphi(s))\eta(s)$. The operator $\mathbf{P}(\varphi)$ was calculated by Mouchrif (1994) in his thesis, using a combination of matched asymptotic expansions and Muskhelishvili's method. His numerical results are given in Appendix F.

Remark. It is possible in the same way to make explicit the influence of the function $\mathbf{K}(s')$: the formula obtained reads

$$\mathbf{K}^{(1)}(s) \equiv \frac{1}{2}\left[\frac{\partial^2\mathbf{K}(s,\delta)}{\partial(\sqrt{\delta})^2}\right]_{\delta(s')\equiv\delta(s),\varphi(s')\equiv\varphi(s)}^{C^*(s)=0,\mathbf{K}(s')\equiv\mathbf{K}(s)}\eta(s)+C^*(s)\mathbf{M}(\varphi(s))\cdot\mathbf{K}(s)\eta(s)$$

$$+\mathbf{N}(\varphi(s))\cdot\mathbf{K}(s)\eta'(s)+\mathbf{P}(\varphi(s))\cdot\mathbf{K}(s)\varphi'(s)\eta(s)+\mathbf{Q}(\varphi(s))\cdot\mathbf{K}'(s)\eta(s)$$

$$+\mathbf{F}(\varphi(s))\cdot PV\int_{\mathscr{F}}\mathbf{Z}(\Omega,s,s')\cdot[(\mathbf{F}^T\cdot\mathbf{F})(\varphi(s'))\cdot\mathbf{K}(s')\eta(s')-(\mathbf{F}^T\cdot\mathbf{F})(\varphi(s))\cdot\mathbf{K}(s)\eta(s)]\,\mathrm{d}s'$$

where $\mathbf{Q}(\varphi)$ is a new universal operator. However we shall not use this formula in the sequel.

6. Comparison with the works of Gao–Rice and Nazarov

Gao and Rice's precursory works in the domain are based on the use of Bueckner's weight functions. The method employed requires the *explicit* knowledge of these functions, which depend upon the geometry of the body and the crack considered, and must therefore be evaluated in each particular case; whence the obligation of considering only special cases. Moreover, it only applies to *coplanar* extensions of *initially plane* cracks. Thus, Rice (1985) studied the case of a semi-infinite crack loaded in pure mode I; his work was extended to arbitrary combinations of modes I, II and III by Gao and Rice[8] (1986). Similarly, Gao and Rice (1987a) considered a tensile penny-shaped crack and their work was extended to arbitrary loadings by Gao (1988). Finally Gao and Rice (1987b) studied the case of an external circular crack, but only in mode I. In all these works, the body considered was infinite. In each special case, the authors established a detailed expression of the term $\mathbf{K}^{(1)}$ of the expansion of the SIFs in powers of the crack advance which fully agrees with eqn (30′). The universality property of the operator $\mathbf{N}(0)$ was noticed by Gao (1988), from the observation that this operator was the same for a semi-infinite crack and a penny-shaped one; his (somewhat qualitative) argument was as follows: "More intuitively it can be imagined that in the very near neighborhood of some special point s along the circular crack front, one would not be able to tell whether the whole crack is a straight line or a circle of some finite radius. The above argument remains valid even for an arbitrary, smoothly curved crack. For this reason, expressions (27) will be generally valid for the local slope effect on the variation of the relative crack dis-

[8] Note however that the hypothesis of coplanar propagation is seldom verified in the presence of mode II.

placements when any smoothly curved crack front gets perturbed with a slope change $d\delta(s)/ds$ at location s along the crack front''. On the other hand, neither Gao nor Rice said anything about the universality of the quantity $\lim_{s'\to s} \mathbf{Z}(\Omega, s, s')(s'-s)^2$, although it was apparent on their results.

In a last paper, Rice (1989) extended the analysis to the case of a plane crack with an arbitrary contour loaded in pure mode I. Again, the results obtained are in full agreement with ours.

Nazarov's (1989) method differs from both ours and that of Gao and Rice; it is based on matched asymptotic expansions (applied to the Neuber–Papkovitch potentials of the problem) combined with the Bueckner–Rice weight function theory. The restrictive hypotheses on the geometry and loading considered are the same as in the work of Rice (1989). The result obtained fully agrees with that of Rice and ours and clearly evidences the universality of the quantity $\lim_{s'\to s} \mathbf{Z}_{\mathrm{I,I}}(\Omega, s, s')(s'-s)^2$.

In conclusion, it appears that the major originality of the present work lies in the fact that it can deal with arbitrarily shaped cracks with arbitrary kinked and curved extensions, which is essential to deal with mixed mode situations. It is remarkable that the method employed, although admittedly quite technical with its three lemmas, makes no great difference between the planar and non-planar cases.

7. Prediction of the propagation path

The expression of the expansion of the SIFs in powers of the crack advance being now available, one can determine the successive positions of the crack by applying some propagation criterion to these SIFs. That which will be considered here is a combination of Goldstein and Salganik's (1974) ''principle of local symmetry'' (PLS) and some energetic criterion of the Griffith type:

$$K_{\mathrm{II}}(s,t) = 0; \quad \mathscr{G}(s,t) \equiv \frac{1-v^2}{E} K_{\mathrm{I}}^2(s,t) + \frac{1+v}{E} K_{\mathrm{III}}^2(s,t)$$
$$\equiv \Lambda_{pq} K_p(s,t) K_q(s,t) = \mathscr{G}_c \ (\forall s, \forall t > 0) \tag{33}$$

where t denotes time, $\mathscr{G}$ the energy release rate and $\mathscr{G}_c$ its critical value. Note that these equations imply, in the limit $t \to 0$, that $K_{\mathrm{II}}^*(s) = 0$ and $\mathscr{G}^*(s) \equiv [(1-v^2)/E]K_{\mathrm{I}}^{*2}(s) + [(1+v)/E]K_{\mathrm{III}}^{*}{}^2(s) = \mathscr{G}_c$. This criterion stipulates that the kink angle depends only on the ratio $K_{\mathrm{II}}/K_{\mathrm{I}}$ and not on K_{III} [since K_{II}^* depends only on K_{I} and K_{II}: see eqns (8) and (9) of Part I]; thus it disregards the phenomenon of crack front segmentation in mode I+III, which precisely consists in oscillations of the kink angle along the crack front of short wavelength about a zero mean value arising from the presence of some mode III. Doing this is acceptable from a macroscopic point of view. The theoretical study of crack front segmentation at the microscopic scale will be the subject of another paper.

7.1. *Introduction of a variable loading*

All expansions of the SIFs obtained in Part I and here were obtained under the assumption of a *constant* loading. It now becomes necessary to introduce a temporal variation of that loading in order to respect the energetic criterion at all instants. It will be assumed here that the loading varies *proportionally*, i.e. through multiplication by some scalar $\mu(t)$ depending only on time, and taking the value 1 at time $t = 0$. The expansion of that scalar in powers of t, which is imposed by the experimenter, does certainly not contain fractional powers and may thus be written as

$$\mu(t) \equiv 1 + \dot{\mu}t + \frac{\ddot{\mu}}{2}t^2 + O(t^3). \quad (34)$$

To obtain the expansion of the SIFs under a variable loading, one may then proceed as follows. First, replace $\varepsilon\eta(s') \equiv \delta(s')$ by its expansion in powers of t in the expressions of the SIFs derived above; this yields the expansion of the SIFs in powers of t under constant loading. Second, using the fact that the solution of an elasticity problem does not depend upon the history of the loading, and thus that the SIFs at the instant t can be obtained by assuming the loading to be constant and equal to its present value, multiply the preceding expansion by $1 + \dot{\mu}t + \ddot{\mu}t^2/2 + O(t^3)$ and reorder the terms to get the final expansion of the SIFs under variable loading.

7.2. *Theoretical determination of the geometric parameters of the crack extension—Case of regular propagation*

Let us first assume that the propagation path is regular. In this case the kink angle $\varphi(s)$ is identically zero and so is also $a^*(s)$. $\mathbf{K}^*(s)$ is identical to $\mathbf{K}(s)$ because $\mathbf{F}(0) = \mathbf{1}$, and $\mathbf{K}^{(1/2)}(s)$ is zero because $a^*(s) = 0$ and $\mathbf{G}(0) = \mathbf{0}$ [see eqn (19) of Part I and Amestoy and Leblond, 1992]. The procedure described above then leads to

$$\mathbf{K}(s,t) = \mathbf{K}(s) + \Bigg\{ \left[\frac{\partial \mathbf{K}(s,\delta)}{\partial \delta} \right]_{\delta(s') \equiv \delta(s)}^{C^*(s)=0} \dot{\delta}(s) + C^*(s)\mathbf{M}(0)\cdot\mathbf{K}(s)\dot{\delta}(s) + \mathbf{N}(0)\cdot\mathbf{K}(s)\dot{\delta}'(s)$$
$$+ PV\int_{\mathscr{F}} \mathbf{Z}(\Omega,s,s')\cdot\mathbf{K}(s')[\dot{\delta}(s') - \dot{\delta}(s)]\,\mathrm{d}s' + \dot{\mu}\mathbf{K}(s) \Bigg\} t + O(t^2). \quad (35)$$

Application of the criterion [eqns (33)] at order $t^0 = 1$ then yields $K_{\mathrm{II}}(s) = 0$ and $[(1-\nu^2)/E]K_{\mathrm{I}}^2(s) + [(1+\nu)/E]K_{\mathrm{III}}^2(s) = \mathscr{G}_c$ (for all s); the second condition here determines the intensity of the loading. To respect the criterion (33) at order $t^1 = t$, one must equate to 0 the terms of this order in the expansions of $K_{\mathrm{II}}(s,t)$ and $\mathscr{G}(s,t)$. The "energetic" part of the criterion yields

$$\Lambda_{pq}K_p(s)\dot{K}_q(s) = 0, \quad p,q = \mathrm{I},\mathrm{III}.$$

Inserting this equation into (35), one gets

$$\Lambda_{pq}K_p(s)\left[\frac{\partial K_q(s,\delta)}{\partial \delta} \right]_{\delta(s') \equiv \delta(s)}^{C^*(s)=0} \dot{\delta}(s)$$
$$+ \Lambda_{pq}K_p(s)\cdot PV\int_{\mathscr{F}} Z_{qr}(\Omega,s,s')K_r(s')[\dot{\delta}(s') - \dot{\delta}(s)]\,\mathrm{d}s' + \dot{\mu}\mathscr{G}_c = 0. \quad (36)$$

The terms $\Lambda_{pq}K_p(s)C^*(s)M_{qr}(0)K_r(s)\dot{\delta}(s)$ and $\Lambda_{pq}K_p(s)N_{qr}(0)K_r(s)\dot{\delta}'(s)$ which should appear here are in fact zero. For the first one, this is because $K_{\mathrm{II}}(s) = 0$, $M_{\mathrm{I,I}}(0) = M_{\mathrm{III,III}}(0) = 0$ and $M_{\mathrm{I,III}}(\varphi) \equiv$

$M_{\mathrm{III,I}}(\varphi) \equiv 0$ (see Appendix F and Section 4 above). For the second one, this is because, as was seen in Section 4, the only non-zero components of $\mathbf{N}(0)$ are $N_{\mathrm{II,III}}(0)$ and $N_{\mathrm{III,II}}(0)$, which appear multiplied by $K_{\mathrm{II}}(s) = 0$.

Equation (36) is an integral equation on the propagation rate $\dot{\delta}(s)$ which allows for its determination *independently of the curvature parameter* $C^*(s)$. Next, the "PLS" part of the criterion yields the value of this parameter in terms of the propagation rate:

$$C^*(s) = -\frac{2}{K_{\mathrm{I}}(s)}\left[\frac{\partial K_{\mathrm{II}}(s,\delta)}{\partial \delta}\right]_{\delta(s') \equiv \delta(s)}^{C^*(s)=0} + \frac{2}{K_{\mathrm{I}}(s)\dot{\delta}(s)}\left\{\frac{2}{2-\nu}K_{\mathrm{III}}(s)\dot{\delta}'(s) - VP\int_{\mathscr{F}} Z_{\mathrm{II},q}(\Omega,s,s')K_q(s')[\dot{\delta}(s')-\dot{\delta}(s)]\,\mathrm{d}s'\right\}. \quad (37)$$

Use has been made here of the properties $M_{\mathrm{II,III}}(\varphi) \equiv 0$, $M_{\mathrm{II,I}}(0) = 1/2$, $N_{\mathrm{II,I}}(0) = 0$, $N_{\mathrm{II,III}}(0) = -2/(2-\nu)$ (see Section 4 above, Appendix F and eqn $(31)_1$). One interesting feature of this formula is the presence of a term proportional to $K_{\mathrm{III}}(s)\dot{\delta}'(s)$ in the expression of $C^*(s)$, which is a typical three-dimensional effect. It can be verified that for a semi-infinite plane crack loaded in mode I+III, this term is the only one which is non-zero. Thus, if K_{III} is uniform, or simply has a constant sign along the front, $C^*(s)$ *takes opposite values on both sides of a local protrusion of the front*; that is, there are deviations from planarity of opposite sign on these two sides. This effect was foreseen by Gao and Rice (1986a), as appears in the following quotation: "As is evident from the exact first order results in eqns (15), mode III loading induces a mode II stress intensity wherever $\mathrm{d}\delta(s)/\mathrm{d}s \neq 0$. This induced K_{II} reverses sign with the change in sign of $\mathrm{d}\delta(s)/\mathrm{d}s$ in going from one side to the other of a localized protrusion. This change in sign of K_{II} is expected to promote deviations from planarity of opposite sense on the two sides of the protrusions". This argument was not fully satisfactory, however, because it was based on the observation of the appearance of a non-zero K_{II}, which was contradictory with the basic assumption made by the authors of coplanar propagation.

7.3. *Theoretical determination of the geometric parameters of the crack extension—Case of a kinked extension*

Let us now consider the case where the kink angle is non-zero. *The propagation rate* $\dot{\delta}(s)$ *is then necessarily zero*. Indeed if it were not, $\sqrt{\delta(s,t)}$ would be proportional to $\sqrt{t}$ for small t, so that the expansion of $\mathscr{G}(s,t)$, prior to introduction of the effect of the variation of the loading, would involve a term proportional to $\sqrt{t}$. Introduction of that effect would not introduce any compensating term, since the expansion (34) of $\mu(t)$ does not involve fractional powers. It would thus be impossible to respect the criterion $\mathscr{G}(s,t) = \mathscr{G}_c$ at all instants. In other words, if there is a kink, the crack advance $\delta(s,t)$ varies necessarily like t^2 and not t for small t. Accounting for this peculiarity and following the procedure indicated in Section 7.1, we get the expansion of the SIFs in the form

$$\mathbf{K}(s,t) = \mathbf{F}(\varphi(s))\cdot\mathbf{K}(s) + \Big\{[\mathbf{G}(\varphi(s))\cdot\mathbf{T}(s) + a^*(s)\mathbf{H}(\varphi(s))\cdot\mathbf{K}(s)]\sqrt{\ddot{\delta}(s)/2} + \dot{\mu}\mathbf{F}(\varphi(s))\cdot\mathbf{K}(s)\Big\}t + O(t^2). \quad (38)$$

Applying the criterion $(33)_1$ at order $t^0 = 1$, one gets

$$K^*_{\mathrm{II}}(s) = F_{\mathrm{II,I}}(\varphi(s))K_{\mathrm{I}}(s) + F_{\mathrm{II,II}}(\varphi(s))K_{\mathrm{II}}(s) = 0, \quad (39)$$

which determines the kink angle $\varphi(s)$ along the crack front as a function of $(K_{\mathrm{II}}/K_{\mathrm{I}})(s)$. The "energetic" part $(33)_2$ of the criterion yields

$$\mathscr{G}^*(s) = \frac{1-\nu^2}{E}K^{*2}_{\mathrm{I}}(s) + \frac{1+\nu}{E}K^{*}_{\mathrm{III}}(s) = \Lambda_{pq}K^*_p(s)K^*_q(s) = \mathscr{G}_c, \quad (40)$$

which fixes the intensity of the loading.

At order $t^1 = t$, one first gets from $(33)_1$, accounting for the properties of the operators $\mathbf{G}(\varphi)$ and $\mathbf{H}(\varphi)$ [see eqn (20) of Part I]:

$$a^*(s) = -\frac{G_{\mathrm{II,I}}(\varphi(s))T_{\mathrm{I}}(s)}{H_{\mathrm{II},q}(\varphi(s))K_q(s)}, \quad q = \mathrm{I,II}. \quad (41)$$

This determines $a^*(s)$ *independently of* $\ddot{\delta}(s)$. This acceleration then follows from eqn $(33)_2$:

$$\ddot{\delta}(s) = 2\left[\frac{\dot{\mu}\mathscr{G}_c}{\Lambda_{pq}F_{pr}(\varphi(s))K_r(s)[G_{qs}(\varphi(s))T_s(s) + a^*(s)H_{qs}(\varphi(s))K_s(s)]}\right]^2. \quad (42)$$

Finally, we shall not write the equations obtained at order t^2, because they are extremely heavy, but just mention that here again, $C^*(s)$ *can be determined independently of* $\dddot{\delta}(s)$; $\dddot{\delta}(s)$ then follows from a *purely local formula*, because what is involved in the integral term appearing in that formula is $\ddot{\delta}(s)$ (which is already known at that stage) and not $\dddot{\delta}(s)$.

It thus appears that it is always possible to *separately* determine the crack advance and the other geometric parameters of the crack extension. In the case of regular propagation, the propagation velocity $\ddot{\delta}(s)$ comes first, and the curvature $C^*(s)$ of the extension follows. If there is a kink, just the opposite occurs: $\varphi(s)$ and $a^*(s)$ come first, then $\ddot{\delta}(s)$; at the next order, $C^*(s)$ comes first and then $\dddot{\delta}(s)$.

7.4. *Numerical determination of the crack path over arbitrary distances*

The theoretical formulae (36), (37), (39), (41), (42) only allow to predict the evolution of the crack in the near future, and provided that the mechanical quantities involved are known. Numerical step-by-step methods can remove these restrictions. Let us consider the case of regular propagation ($\varphi(s') \equiv 0$, $a^*(s') \equiv 0$) for instance. The initial geometry being known, one can compute the initial SIFs. The derivatives $[\partial K_p(s,\delta)/\partial\delta]^{C^*(s)=0}_{\delta(s')\equiv\delta(s)}$ can be evaluated by adding a small, straight ($C^*(s) = 0$) extension with uniform length ($\delta(s') \equiv Cst.$) and comparing the SIFs along the front of the extended crack with the initial ones. Finally, using eqn (30′), the components of the non-universal operator $\mathbf{Z}(\Omega, s, s')$ can be determined by creating a small protrusion of the crack front in the vicinity of the point s' and examining the resulting effect on the SIFs at the point s. The fact that the quantities $\lim_{s'\to s} Z_{pq}(\Omega, s, s')(s'-s)^2$ are universal and known (see eqns (B4) of Appendix B) should be a great help there. All coefficients in the left-hand side of eqn (36) being then known, discretization and numerical solution of this equation yield the propagation rate $\dot{\delta}(s)$ along the front. The curvature $C^*(s)$ then follows from eqn (37). It only remains to numerically extend the crack according to the parameters determined (remeshing operation) to proceed to the next step.

8. A few simple applications

We shall finally envisage some simple applications which do not require the calculation of the non-universal operator $\mathbf{Z}(\Omega, s, s')$ for the specific geometry considered as a prerequisite (see Leblond et al. (1996) for an application that does require this calculation). Just as in the preceding section, we adopt the point of view that the PLS applies even in the presence of mode III on a macroscopic scale.

8.1. *Configurational stability of the front of a propagating crack*

Following the works of Gao and Rice quoted above, we shall now examine the question of the stability of the "fundamental" configuration of the front vs in-plane perturbations within the new tangent plane to the crack.[9] For a given sinusoidal perturbation $\eta(s)$ of the crack front, stability will prevail if the maxima of $\eta(s)$ coincide with the minima of $\mathscr{G}(s, \varepsilon)$ (and vice versa), and instability will prevail if the maxima of these quantities coincide. We shall distinguish between the case of regular propagation and that of a kinked extension.

In the case of regular propagation ($\varphi(s) \equiv 0 \Rightarrow K_{\mathrm{II}}(s) \equiv 0$), we shall follow the works of Rice and Gao quoted above and consider the cases of a semi-infinite crack and a penny-shaped one. For a semi-infinite crack loaded in mode I+III, it was concluded by Gao and Rice (1986) that the straight configuration of the front was stable vs in-plane sinusoidal perturbations of all wavelengths. On the other hand, for a penny-shaped crack loaded in torsion (pure mode III), Gao (1988) found that the circular configuration of the front was unstable vs in-plane sinusoidal perturbations provided that there were less than $(8-5\nu)/(2+\nu)$ wavelengths in the perimeter of the crack. However, these conclusions were obtained assuming the propagation to be coplanar, which is generally wrong because, as discussed above, $C^*(s)$ is generally non-zero in the presence of mode III [see eqn (37)]. Therefore the following question naturally arises: will these conclusions be affected by non-coplanarity?

In eqn (30′), the components $K_{\mathrm{I}}^{(1)}(s)$ and $K_{\mathrm{III}}^{(1)}(s)$ happen to be independent of the value of $C^*(s)$, because of the properties $M_{\mathrm{I,I}}(0) = M_{\mathrm{III,III}}(0) = 0$, $M_{\mathrm{I,III}}(\varphi) \equiv M_{\mathrm{III,I}}(\varphi) \equiv 0$ (see Appendix F and

[9] Note that the kind of stability discussed here deals with the shape of the crack front in the $Ox_1^*x_3$ plane, where $(Ox_1^*x_2^*x_3)$ denotes the frame "adapted" to the crack extension (see Fig. 3 of Part I); it should not be confused with the sort of stability envisaged by Cotterell and Rice (1980) and in Part I, which considers the shape of the crack in the Ox_1x_2 plane.

Section 4 above). It follows that the same is true of $\mathscr{G}(s,t) = [(1-\nu^2)/E]K_{\rm I}^2(s,t) \ +[(1+\nu)/E]K_{\rm III}^2(s,t)$ at order $t^1 = t$, so that *accounting for the non-zero value of* $C^*(s)$ *does not change Gao and Rice's conclusions in any way*.

Now consider the case where $K_{\rm II}(s) \not\equiv 0 \Rightarrow \varphi(s) \not\equiv 0$, for instance a tunnel-crack loaded in mode I+II through uniform stresses σ_{11}^∞, σ_{12}^∞, σ_{22}^∞ exerted at infinity. The expansion of $\mathscr{G}(s,\varepsilon)$ at order $\sqrt{\varepsilon}$ then reads, by eqns (17) and (19) of Part I:

$$\mathscr{G}(s,\varepsilon) = \Lambda_{pq}K_p^*K_q^* + 2\Lambda_{pq}K_p^*[G_{qr}(\varphi)T_r + a^*H_{qr}(\varphi)K_r]\sqrt{\varepsilon\eta(s)} + O(\varepsilon),$$

where $K_p^* = F_{ps}(\varphi)K_s$ and a^* is given by eqn (41). (Certain dependencies upon the position s along the crack front disappear here because of translational invariance in the direction of that front.) It follows that *the fact that the maxima of* $\eta(s)$ *(or* $\delta(s)$*) and* $\mathscr{G}(s,\varepsilon)$ *coincide or not is totally independent of the wavelength of the perturbation and depends only on the sign of the quantity* $\Lambda_{pq}K_p^*[G_{qr}(\varphi)T_r + a^*H_{qr}(\varphi)K_r] \equiv [(1-\nu^2)/E]K_{\rm I}^*[G_{\rm I,I}(\varphi)T_{\rm I} + a^*H_{{\rm I}r}(\varphi)K_r]$: stability prevails if this quantity is negative. Now $K_{\rm I}^*$ must be positive for the crack extension to be opened, $G_{\rm I,I}(\varphi)$ is positive (see Amestoy, 1987) and $H_{{\rm I}r}(\varphi)K_r$ happens to be very small for that value of φ which ensures the condition $K_{\rm II}^* = 0$. Hence the condition of stability simply reads

$$T_{\rm I} < 0.$$

(Note, again, that although this condition is exactly the same as that found by Cotterell and Rice, 1980, we are not dealing here with the same kind of stability.) Hence, if the stresses imposed at infinity are such that $T_{\rm I} \equiv \sigma_{11}^\infty - \sigma_{22}^\infty > 0 \Leftrightarrow \sigma_{11}^\infty > \sigma_{22}^\infty$, one should observe, after kinking, unstable undulations of the crack front in the new tangent plane to the crack. (This conclusion is arrived at by considering only the first two terms, proportional to $\varepsilon^0 = 1$ and $\varepsilon^{1/2}$, of the expansion of $\mathscr{G}(s,\varepsilon)$ in powers of ε. This means that stability may prevail again once the crack has propagated over a sufficient distance and the third term, proportional to $\varepsilon^1 = \varepsilon$, becomes important.)

8.2. *Bifurcation of the propagation path*

We now consider the problem of *uniqueness* of the propagation path, as determined following the procedures described above. Again, we shall distinguish between the case of regular propagation and that of a kinked extension.

In the case of regular propagation, we have seen that the propagation rate is determined by the *integral equation* (36). It follows that uniqueness of $\dot{\delta}(s)$ is not guaranteed. For instance, Leblond et al. (1996) have considered the case of a tensile (mode I) tunnel-crack. They have shown that there is a sinusoidal bifurcation mode which is symmetric with respect to the axis of the tunnel-crack and the wavelength of which is $6.793a$ where a is the half-width of the crack. Another interesting example is provided by the penny-shaped crack in torsion (pure mode III) considered by Gao (1988). For that configuration, there is a possible sinusoidal bifurcation mode, with $(8-5\nu)/(2+\nu)$ wavelengths in the perimeter of the crack. However, for this bifurcation mode to exist, this number must be an integer, which implies that ν must be equal to $(8-2n)/(n+5)$ where n is an integer; in practice, since $0 < \nu < 1/2$, this imposes $\nu = 1/4$ (with $n = 3$), which will never be satisfied *exactly*. One sees that the essential difference between the two examples is that the crack front is *finite* in the second one; this creates a "quantization" of wavelengths which rules out the critical bifurcation wavelength.

If there is a kink, we have seen that all geometric parameters of the crack extension are determined by purely local equations (integrals do remain, but they do not involve unknowns). It follows that *any bifurcation is precluded in the presence of mode* II, at least if the expansion of the SIFs is limited to order t^2, as was done here.

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Appendix A

The aim of this Appendix is to establish Lemma 1 of the text.

We shall begin by re-interpreting the components of the operator $\mathbf{Z}(\Omega, R, s, s')$ in a simpler way. Our primary interpretation is provided by eqn (14) of the text. Remember that $K_r^{(i)}(\Omega, s', M)$ is the r-th SIF generated at the point s' of the initial crack front by a unit point force exerted on the point M in the direction $\mathbf{E}_i \equiv \partial \mathcal{O}\mathbf{M}/\partial X_i$, with $\mathbf{u} = \mathbf{0}$ on $\partial\Omega_u$ and $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$ on $\partial\Omega_t$—the problem thus defined will be called Problem A. The vector $\mathbf{K}_r(\Omega, s', M) \equiv [K_r^{(1)}, K_r^{(2)}, K_r^{(3)}](\Omega, s', M)$ being then re-interpreted as a displacement at the point M, $\{\mathbf{L}_M \cdot \mathbf{K}_r(\Omega, s', M)\}_{\partial\mathcal{S}(s,R)}$ is the traction field, on the boundary of the sphere $\mathcal{S}(s, R)$ of center s and radius R, corresponding to the displacement field $\{\mathbf{K}_r(\Omega, s', M)\}$. Finally $\mathcal{L}_p[R, \mathbf{C}, \Gamma; \{\mathbf{L}_M \cdot \mathbf{K}_r(\Omega, s', M)\}_{\partial\mathcal{S}(s,R)}]$ is the p-th SIF at the point s of the initial crack front generated by these tractions.

Let (s', r') denote a point on the crack lips (r' representing the first polar coordinate in the plane orthogonal to the crack front at the point s'). For the Problem A just defined, the discontinuity $[\![u_j^{(i)}]\!](s', r')$ of the j-th component, in the general, fixed basis $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ of the displacement $\mathbf{u}^{(i)}$ at the point (s', r') is given by

$$[\![u_j^{(i)}]\!](s', r') = 2\Delta_{jp}^{-1}(s')\Lambda_{pq}K_q^{(i)}(\Omega, s', M)\sqrt{r'} + O(r') \tag{A1}$$

where $\mathbf{\Lambda}$ is the matrix defined by eqn (10) of the text and $\mathbf{\Delta}^{-1}(s')$ a matrix characterizing the orientation of the local basis "adapted" to the crack at the point s' of the (initial) crack front with respect to the general one. We then define a *unit doublet*[10] of direction $\mathbf{E}_j$, applied on the point s' of the (initial) crack front, as the limit, for $r' \to 0$, of a loading consisting of two opposite forces parallel to $\mathbf{E}_j$, with intensity $1/\sqrt{r'}$, applied on the points $(s', r')^{\pm}$ of the upper and lower lips of the crack. Let us define Problem B as that where such a doublet is applied on s', $\partial\Omega_u$ being clamped and $\partial\Omega_t$ free of tractions, and $v_i^{(j)}(\Omega, M, s')$ as the i-th component of the resulting displacement at the point M (again, the argument Ω underlines the dependence upon the whole geometry of the body and crack considered via the boundary conditions). Applying Betti's theorem to Problems A and B, we get

[10] A doublet, which is necessarily exerted on a point of the crack front, should not be confused with a dipole (see Part I, Section 3), which can be applied anywhere in the body.

$$v_i^{(j)}(\Omega, M, s') = 2\Delta_{jp}^{-1}(s')\Lambda_{pq}K_q^{(i)}(\Omega, s', M) \Rightarrow 2\Lambda_{pq}K_q^{(i)}(\Omega, s', M) = \Delta_{pj}(s')v_i^{(j)}(\Omega, M, s')$$

or, in vectorial form:

$$2\Lambda_{pq}\mathbf{K}_q(\Omega, s', M) = \Delta_{pj}(s')\mathbf{v}^{(j)}(\Omega, M, s').$$

It follows from that result that eqn (14) of the text may be rewritten as

$$Z_{pq}(\Omega, R, s, s') = \Delta_{qj}(s')\mathscr{L}_p[R, \mathbf{C}, \Gamma; \{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega, M, s')\}_{\partial\mathscr{S}(s,R)}]. \tag{A2}$$

According to this equation, the components of the operator $\mathbf{Z}(\Omega, R, s, s')$ are connected, through some multiplicative factors $\Delta_{qj}(s')$, to the SIFs generated at the point s of the initial crack front by the application, on the boundary of the sphere $\mathscr{S}(s, R)$, of the traction fields $\{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega, M, s')\}$ ($j = 1, 2, 3$). These traction fields correspond to the displacement fields $\{\mathbf{v}^{(j)}(\Omega, M, s')\}$ resulting from the application of unit doublets at the point s' of the initial crack front, $\partial\Omega_u$ being clamped and $\partial\Omega_t$ free of tractions. This interpretation of the components $Z_{pq}(\Omega, R, s, s')$ is more appealing than the preceding one resulting from eqn (14) of the text, because the formation of the tractions from the displacements is more natural: $\mathbf{v}^{(j)}(\Omega, M, s')$ is directly defined as a displacement at the point M, whereas $\mathbf{K}_r(\Omega, s', M)$ initially represented a set of SIFs at the point s' of the initial crack front and had to be re-interpreted as a displacement at M before one could take the corresponding traction.

We shall have to distinguish between the cases $R < D(s, s')$ and $R > D(s, s')$. Only the first one will be considered here.

Let us consider, in addition to Problem B defined above, Problem C where the exterior of the sphere $\mathscr{S}(s, R)$ (including the doublet at s', since $D(s, s') > R$) is eliminated while the traction field $\{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega, M, s')\}$ of Problem B is preserved on $\partial\mathscr{S}(s, R)$. The SIFs are obviously the same in the two problems. Let $K_p^{(j)}(\Omega, s, s')$ denote the p-th SIF at s in Problem B. (This SIF is obviously not a function of R, since this parameter does not appear in the definition of that problem.) The p-th SIF at s in Problem C is, by definition, $\mathscr{L}_p[R, \mathbf{C}, \Gamma; \{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega, M, s')\}_{\partial\mathscr{S}(s,R)}]$. Therefore,

$$K_p^{(j)}(\Omega, s, s') = \mathscr{L}_p[R, \mathbf{C}, \Gamma; \{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega, M, s')\}_{\partial\mathscr{S}(s,R)}]$$

which implies, by eqn (A2), that

$$\forall R < D(s, s'): \quad Z_{pq}(\Omega, R, s, s') = \Delta_{qj}(s')K_p^{(j)}(\Omega, s, s') \equiv Z_{pq}(\Omega, s, s') \tag{A3}$$

where it clearly appears that $Z_{pq}(\Omega, R, s, s')$ is in reality independent of R for $R < D(s, s')$ (its value in such conditions is simply noted $Z_{pq}(\Omega, s, s')$). This establishes Lemma 1.

Appendix B

The goal of this Appendix is to prove Lemma 2 of the text.

We begin by studying the second case distinguished in Appendix A, where $R > D(s, s')$. Just as in the first one, the $Z_{pq}(\Omega, R, s, s')$ are connected to the SIFs at the point s of the initial crack front arising from the traction fields $\{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega, M, s')\}$ ($j = 1, 2, 3$) exerted on the boundary of the sphere $\mathscr{S}(s, R)$ (Problem C), and the traction field $\{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega, M, s')\}_{\partial\mathscr{S}(s,R)}$ arises from the application of a unit doublet in the direction $\mathbf{E}_j$ on the point s' of the front, $\partial\Omega_u$ being clamped and $\partial\Omega_t$

free of tractions (Problem B). The novelty is that the point s' now lies *within* $\mathscr{S}(s,R)$. One must then take care of the fact that in Problem C, the loading consists only of the surface tractions (on $\partial\mathscr{S}(s,R)$) $\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega,M,s')$, *the doublet at* s' *being eliminated.* Let us therefore define Problems D and E in the following way: in Problem D, the exterior of $\mathscr{S}(s,R)$ is eliminated while the traction field $\{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega,M,s')\}_{\partial\mathscr{S}(s,R)}$ *and* the doublet at s' are preserved; in Problem E, the sole doublet is applied on the point s', $\partial\mathscr{S}(s,R)$ being free of tractions. With obvious notations, one then has $\mathrm{C} \equiv \mathrm{D}-\mathrm{E}$. Now, as already mentioned, the p-th SIF at s in Problem C is, by definition, $\mathscr{L}_p[R,\mathbf{C},\Gamma;\{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega,M,s')\}_{\partial\mathscr{S}(s,R)}]$. In Problem D, it is the same as in Problem B, that is, $K_p^{(j)}(\Omega,s,s')$. Finally we shall denote $K_p^{(j)}(R,s,s')$ that in Problem E; this notation is coherent with the notation $K_p^{(j)}(\Omega,s,s')$, the body Ω being here the sphere $\mathscr{S}(s,R)$, free of boundary tractions. Therefore

$$\mathscr{L}_p[R,\mathbf{C},\Gamma;\{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega,M,s')\}_{\partial\mathscr{S}(s,R)}] = K_p^{(j)}(\Omega,s,s')-K_p^{(j)}(R,s,s')$$

which implies, by eqn (A2), that

$$\forall R > D(s,s'):\quad Z_{pq}(\Omega,R,s,s') = \Delta_{qj}(s')[K_p^{(j)}(\Omega,s,s')-K_p^{(j)}(R,s,s')]. \tag{B1}$$

We now come to the proof of Lemma 2 itself. Its principle is to introduce, for every R_0 such that $\mathscr{S}(s,R) \subset \mathscr{S}(s,R_0) \subset \Omega$ and every $s' \in \mathscr{S}(s,R_0)$, some operator $\mathbf{Z}(R_0,R,s,s')$ identical to $\mathbf{Z}(\Omega,R,s,s')$ except for the replacement of the body Ω by the sphere $\mathscr{S}(s,R_0)$, free of boundary tractions. We shall then show that the quantity $\lim_{s'\to s}\mathbf{Z}(\Omega,s,s')(s'-s)^2$ can be evaluated by replacing $\mathbf{Z}(\Omega,s,s')$ by $\mathbf{Z}(R_0,s,s')$ (quantity defined as the constant value of $\mathbf{Z}(R_0,R,s,s')$ for $R < D(s,s')$, just as for $\mathbf{Z}(\Omega,R,s,s')$ and $\mathbf{Z}(\Omega,s,s')$). As a result, the problem will no longer be posed on an arbitrary body Ω but on the sphere $\mathscr{S}(s,R_0)$. We shall then use the homogeneity properties of the quantity $\lim_{s'\to s}\mathbf{Z}(R_0,s,s')(s'-s)^2$ and shrink R_0 to 0 in order to establish its universality. Finally, we shall refer to the work of Gao and Rice (1986) devoted to the special case of a semi-infinite crack with a slightly wavy front to identify the values of the components of this universal quantity and check that they are finite.

Equation (B1), used with $\Omega \equiv \mathscr{S}(s,R_0)$, yields

$$\forall R > D(s,s'):\quad Z_{pq}(R_0,R,s,s') = \Delta_{qj}(s')[K_p^{(j)}(R_0,s,s')-K_p^{(j)}(R,s,s')].$$

Taking the difference with eqn (B1), we get

$$Z_{pq}(\Omega,R,s,s')-Z_{pq}(R_0,R,s,s') = \Delta_{qj}(s')[K_p^{(j)}(\Omega,s,s')-K_p^{(j)}(R_0,s,s')] \equiv Z_{pq}(\Omega,R_0,s,s'),$$

where we have re-used eqn (B1) with $R \equiv R_0$ ($>D(s,s')$). *A priori*, this result only applies for $R > D(s,s')$, but using eqn (A3) for Ω and $\mathscr{S}(s,R_0)$, one sees that it is also valid if $R < D(s,s')$. Thus, the difference $\mathbf{Z}(\Omega,R,s,s')-\mathbf{Z}(R_0,R,s,s')$ is independent of R, no matter whether R is smaller or greater than $D(s,s')$. Furthermore, its components, i.e. the quantities $Z_{pq}(\Omega,R_0,s,s')$, are tied through some multiplicative factors to the SIFs at s generated by the application of traction fields of the form $\{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega,M,s')\}$ on $\partial\mathscr{S}(s,R_0)$, and these traction fields themselves arise from the application of unit doublets at s', $\partial\Omega_u$ being clamped and $\partial\Omega_t$ free of tractions. Now, when the point s' moves along the crack front, even in the vicinity of s, the traction field on $\partial\mathscr{S}(s,R_0)$ generated by the doublet applied on s' varies continuously and remains finite, and so do also the SIFs at s resulting from these tractions (the doublet at s' being eliminated). It follows that

$Z_{pq}(\Omega, R_0, s, s')$ is a continuous function of s', taking a finite value for $s' = s$. In brief, we have shown at that stage that *the difference* $\mathbf{Z}(\Omega, R, s, s') - \mathbf{Z}(R_0, R, s, s')$ *is independent of* R, *continuous with respect to* s' *and finite for* $s' = s$.

This difference, being independent of R, is equal to its limit for $R \to 0$, i.e. $\mathbf{Z}(\Omega, s, s') - \mathbf{Z}(R_0, s, s')$ by definition of $\mathbf{Z}(\Omega, s, s')$ and $\mathbf{Z}(R_0, s, s')$. Hence, by what precedes,

$$\lim_{s' \to s} [\mathbf{Z}(\Omega, s, s') - \mathbf{Z}(R_0, s, s')](s'-s)^2 = 0 \Rightarrow \lim_{s' \to s} \mathbf{Z}(\Omega, s, s')(s'-s)^2 = \lim_{s' \to s} \mathbf{Z}(R_0, s, s')(s'-s)^2 \tag{B2}$$

(the two last limits being possibly infinite).

We shall now show that the quantity $\lim_{s' \to s} \mathbf{Z}(R_0, s, s')(s'-s)^2$ is universal and therefore, by eqn (B2), that the same is true of $\lim_{s' \to s} \mathbf{Z}(\Omega, s, s')(s'-s)^2$. We set

$$\lim_{s' \to s} \mathbf{Z}(R_0, s, s')(s'-s)^2 \equiv \mathscr{M}(R_0, \mathbf{C}, \Gamma) \tag{B3}$$

(note that the parameters a^*, C^*, $\varepsilon\eta$ and η'/η characterizing the geometry of the crack extension are absent here, since the problems which define $\mathbf{Z}(R_0, s, s')$ only involve the *initial* crack).

Let us study the homogeneity properties of the function $\mathscr{M}(R_0, \mathbf{C}, \Gamma)$. If all distances and displacements are multiplied by some positive factor λ, the stresses remain unchanged. The equilibrium equations imply that the body forces are divided by λ, so that point forces, which are homogeneous to some body force times some volume, are multiplied by λ^2. A unit doublet applied at the distance r' ($\to 0$) of the (initial) crack front, consisting of point forces of intensity $1/\sqrt{r'}$, is now applied at the distance $\lambda r'$ of the crack front and its intensity is $\lambda^2/\sqrt{r'}$. To transform it into a unit doublet involving forces of intensity $1/\sqrt{\lambda r'}$, one must, using linearity, re-multiply the displacements and stresses by $\lambda^{-5/2}$. The net result is that while distances are multiplied by λ, displacements are multiplied by $\lambda \cdot \lambda^{-5/2} = \lambda^{-3/2}$ and stresses by $\lambda^{-5/2}$. In this operation, the SIFs $K_p^{(j)}(R_0, s, s')$, which are limits of certain stress components times the square root of a vanishingly small distance, are multiplied by $\lambda^{-5/2} \cdot \lambda^{1/2} = \lambda^{-2}$. Using eqn $(\mathrm{A3})_2$ and noting that the matrix $\mathbf{\Delta}(s')$ introduced in eqn (A1) remains invariant while distances are multiplied by λ, one sees that $Z_{pq}(R_0, s, s')$ is also multiplied by λ^{-2}. It immediately follows from there and the definition (B3) of the function $\mathscr{M}(R_0, \mathbf{C}, \Gamma)$ that it verifies the following property: $\mathscr{M}(\lambda R_0, \mathbf{C}/\lambda, \Gamma/\lambda) = \mathscr{M}(R_0, \mathbf{C}, \Gamma)$. Using this relation with $\lambda = 1/R_0$ and $R_0 \to 0$, we get $\mathscr{M}(R_0, \mathbf{C}, \Gamma) = \mathscr{M}(1, \mathbf{0}, 0)$, which shows that $\lim_{s' \to s} \mathbf{Z}(R_0, s, s')(s'-s)^2$ does not depend on any geometric parameter whatsoever, i.e. that it is a universal quantity.

Gao and Rice (1986) have studied the special case of a semi-infinite planar crack with a slightly wavy front. For a crack advance $\varepsilon\eta(s')$ which is zero in some open interval $\mathscr{I}$ containing the point s, they find that

$$K_{\mathrm{I}}^{(1)}(s) = \frac{1}{2\pi} \int_{\mathbb{R}-\mathscr{I}} \frac{K_{\mathrm{I}}(s')\eta(s')\,\mathrm{d}s'}{(s'-s)^2}; \quad K_{\mathrm{II}}^{(1)}(s) = \frac{1}{2\pi} \frac{2-3\nu}{2-\nu} \int_{\mathbb{R}-\mathscr{I}} \frac{K_{\mathrm{II}}(s')\eta(s')\,\mathrm{d}s'}{(s'-s)^2};$$

$$K_{\mathrm{III}}^{(1)}(s) = \frac{1}{2\pi} \frac{2+\nu}{2-\nu} \int_{\mathbb{R}-\mathscr{I}} \frac{K_{\mathrm{III}}(s')\eta(s')\,\mathrm{d}s'}{(s'-s)^2}.$$

Now it is easy to see, using eqns (8), (9) and (13) of the text and Lemma 1, that for such a crack advance,

$$\mathbf{K}^{(1)}(s) = \int_{\mathbb{R}-\mathscr{I}} \mathbf{Z}(\Omega, s, s') \cdot \mathbf{K}(s')\eta(s')\, \mathrm{d}s'.$$

Comparison with Gao and Rice's formulae show that

$$\lim_{s'\to s} Z_{\mathrm{I,I}}(\Omega, s, s')(s'-s)^2 = \frac{1}{2\pi}; \quad \lim_{s'\to s} Z_{\mathrm{II,II}}(\Omega, s, s')(s'-s)^2 = \frac{1}{2\pi}\frac{2-3\nu}{2-\nu};$$

$$\lim_{s'\to s} Z_{\mathrm{III,III}}(\Omega, s, s')(s'-s)^2 = \frac{1}{2\pi}\frac{2+\nu}{2-\nu}, \quad \text{(B4)}$$

other components being zero. This shows that these limits are finite and finishes the proof of Lemma 2.

Remark. The preceding results clearly evidence the difference of the behaviors of $\mathbf{Z}(\Omega, R, s, s')$ (R being fixed) and $\mathbf{Z}(\Omega, s, s')$ for $s' \to s$. Indeed we have seen above that $\mathbf{Z}(\Omega, R_0, s, s')$ is a continuous function of s' taking some finite value for $s' = s$, and the same is of course true of $\mathbf{Z}(\Omega, R, s, s')$. On the contrary, Lemma 2 stipulates that $\mathbf{Z}(\Omega, s, s')$ diverges like $(s'-s)^{-2}$ for $s' \to s$. This clearly shows that the convergence of $\mathbf{Z}(\Omega, R, s, s')$ towards $\mathbf{Z}(\Omega, s, s')$ for $R \to 0$ cannot be uniform with respect to s' for s' close to s.

Appendix C

In this Appendix, we establish Lemma 3 of the text.

Let us first note that it is sufficient to prove that for any function $f(s')$ which is $O((s'-s)^2)$ for $s' \to s$,

$$\lim_{R\to 0} \int_{\mathscr{F}} Z_{pq}(\Omega, R, s, s') f(s')\, \mathrm{d}s' = \int_{\mathscr{F}} Z_{pq}(\Omega, s, s') f(s')\, \mathrm{d}s'.$$

According to eqns (A3) and (B1), the integrands here differ only for $D(s, s') < R$, the difference being then $\Delta_{qj}(s') K_p^{(j)}(R, s, s') f(s')$. Hence the problem is reduced to showing that the integral

$$\int_{\mathscr{F} \cap \mathscr{S}(s,R)} \Delta_{qj}(s') K_p^{(j)}(R, s, s') f(s')\, \mathrm{d}s'$$

tends to zero with R.

Let us expand the function $f(s')$ to the third order with respect to $s'-s$ in the vicinity of the point s:

$$f(s') = \tfrac{1}{2} f''(s)(s'-s)^2 + \tfrac{1}{6} f'''(s)(s'-s)^3$$

(it can be checked *a posteriori* that pursuing the expansion up to higher orders would not introduce any change in the conclusion to be reached). Let us set

$$\int_{\mathscr{F}\cap\mathscr{S}(s,R)} \Delta_{qj}(s')K_p^{(j)}(R,s,s')f(s')\,\mathrm{d}s'$$
$$= \int_{\mathscr{F}\cap\mathscr{S}(s,R)} \Delta_{qj}(s')K_p^{(j)}(R,s,s')\frac{f''(s)}{2}(s'-s)^2\left[1+\frac{f'''(s)}{3f''(s)}(s'-s)\right]\mathrm{d}s' \tag{C1}$$
$$\equiv f''\mathscr{N}(R,\mathbf{C},\Gamma,f'''/f'').$$

(Remember that the various parameters are implicitly taken at the point s, as usual, in the last expression.) We have seen in Appendix B that the quantities $K_p^{(j)}(R,s,s')$ are positively homogeneous of degree -2. If we decide, while multiplying all distances by a positive factor λ, to keep the function f unchanged, the left-hand side of eqn (C1) is multiplied by λ^{-1} in the process. Since f'' is multiplied by λ^{-2}, it follows that $\mathscr{N}$ is multiplied by λ:

$$\mathscr{N}(\lambda R,\mathbf{C}/\lambda,\Gamma/\lambda,f'''/(\lambda f'')) = \lambda\mathscr{N}(R,\mathbf{C},\Gamma,f'''/f'').$$

Applying this formula with $\lambda = 1/R$ and $R\to 0$, we get

$$\mathscr{N}(R,\mathbf{C},\Gamma,f'''/f'') = R\mathscr{N}(1,R\mathbf{C},R\Gamma,Rf'''/f'') = R[\mathscr{N}(1,\mathbf{0},0,0)+O(R)]. \tag{C2}$$

Thus, unless $\mathscr{N}(1,\mathbf{0},0,0)$ is infinite, $\mathscr{N}(R,\mathbf{C},\Gamma,f'''/f'')$ is $O(R)$ and goes to 0 with R, which is the desired result. Therefore, the problem is reduced to showing that $\mathscr{N}(1,\mathbf{0},0,0)$ is finite.

Up to the factor f'', this expression is nothing else than the integral we started with, $\int_{\mathscr{F}\cap\mathscr{S}(s,R)}\Delta_{qj}(s')K_p^{(j)}(R,s,s')f(s')\,\mathrm{d}s'$, for certain values of the geometric parameters. The problem is to show that this integral is convergent, which is not a triviality because of the singular behavior of $K_p^{(j)}(R,s,s')$ for $s'\to s$ and $s'\to\partial\mathscr{S}(s,R)$.

With regard to the first case, $s'\to s$, Lemma 2 (applied with $\Omega\equiv\mathscr{S}(s,R)$) implies that $K_p^{(j)}(R,s,s')$ behaves like $(s'-s)^{-2}$. Since $f(s')$ is $O((s'-s)^2)$, the integrand remains finite for $s'\to s$ and there is no problem of convergence.

With regard to the second case, we shall first exchange the roles of s and s' as points of application of the doublet and observation of the SIFs. Let us call Problem E, as before, that where a unit doublet is applied on the point s' in the direction $\mathbf{E}_j$, $\partial\mathscr{S}(s,R)$ being free of tractions. Recall that such a "doublet" consists of two opposite forces parallel to $\mathbf{E}_j$ with intensity $1/\sqrt{r'}$ applied on the points $(s',r')^{\pm}$ of the crack lips, with $r'\to 0$. The discontinuity of the i-th component of the displacement at the point (s,r) of the crack surface is then $2\Delta_{iq}^{-1}(s)\Lambda_{qp}K_p^{(j)}(R,s,s')\sqrt{r}+O(r)$ [see eqn (A1)]. Let us call Problem F that where a unit doublet is applied on the point s in the direction $\mathbf{E}_i$, $\partial\mathscr{S}(s,R)$ being free of tractions. The resulting discontinuity of the j-th component of the displacement at the point (s',r') of the crack surface is $2\Delta_{jm}^{-1}(s')\Lambda_{mn}K_n^{(i)}(R,s',s)\sqrt{r'}+O(r')$. Applying Betti's theorem to these two problems, we get

$$2\Delta_{iq}^{-1}(s)\Lambda_{qp}K_p^{(j)}(R,s,s') = 2\Delta_{jm}^{-1}(s')\Lambda_{mn}K_n^{(i)}(R,s',s)$$
$$\Rightarrow K_p^{(j)}(R,s,s') = \Lambda_{pq}^{-1}\Delta_{qi}(s)\Delta_{jm}^{-1}(s')\Lambda_{mn}K_n^{(i)}(R,s',s). \tag{C3}$$

The matrices $\mathbf{\Delta}(s)$ and $\mathbf{\Delta}^{-1}(s')$ being regular functions of their argument, eqn (C3) shows *that the behavior of* $K_p^{(j)}(R,s,s')$ *for* $s'\to\partial\mathscr{S}(s,R)$ *is identical to that of* $K_n^{(i)}(R,s',s)$. Remember that the latter quantity is a SIF generated at the point s' of the initial crack front (instead of s) by applying a unit doublet on the point s (instead of s').

It is shown in Appendix D that near a free surface like $\partial\mathscr{S}(s,R)$, the SIFs of a surface crack behave like σ^{β}, where σ denotes the distance from the point of observation of the SIFs to the free surface along the crack front, and β an exponent which is greater than -1. Thus, $K_n^{(i)}(R,s',s)$ and $K_p^{(j)}(R,s,s')$ behave like $[R-D(s,s')]^{\beta}$ for $s'\to\partial\mathscr{S}(s,R)$ with $\beta>-1$, and the convergence of the integral $\int_{\mathscr{F}\cap\mathscr{S}(s,R)}\Delta_{qj}(s')K_p^{(j)}(R,s,s')f(s')\,\mathrm{d}s'$ follows from there.

Remark. The "reciprocity" relation (C3) was established for $\Omega\equiv\mathscr{S}(s,R)$, $\partial\mathscr{S}(s,R)$ being free of tractions, but since its proof used only Betti's theorem, it holds for an arbitrary body subjected to arbitrary boundary conditions. Combining it with eqn $(A3)_2$, one then finds that

$$\Lambda_{qp}Z_{pm}(\Omega,s,s')=\Lambda_{mn}Z_{nq}(\Omega,s',s)\Rightarrow\mathbf{\Lambda}\cdot\mathbf{Z}(\Omega,s,s')=[\mathbf{\Lambda}\cdot\mathbf{Z}(\Omega,s',s)]^{T}. \tag{C4}$$

Writing explicitly the components of this tensorial relation using the definition (10) of the matrix $\mathbf{\Lambda}$, one gets

$$\begin{gathered}Z_{\mathrm{I,I}}(\Omega,s,s')=Z_{\mathrm{I,I}}(\Omega,s',s);\quad Z_{\mathrm{II,II}}(\Omega,s,s')=Z_{\mathrm{II,II}}(\Omega,s',s);\\ Z_{\mathrm{III,III}}(\Omega,s,s')=Z_{\mathrm{III,III}}(\Omega,s',s);\quad Z_{\mathrm{I,II}}(\Omega,s,s')=Z_{\mathrm{II,I}}(\Omega,s',s);\\ (1-\nu)Z_{\mathrm{I,III}}(\Omega,s,s')=Z_{\mathrm{III,I}}(\Omega,s',s);\quad (1-\nu)Z_{\mathrm{II,III}}(\Omega,s,s')=Z_{\mathrm{III,II}}(\Omega,s',s).\end{gathered} \tag{C4$'$}$$

All of these relations are found to be satisfied in the special cases investigated by Rice (1985), Gao and Rice (1986, 1987a, b) and Gao (1988). Also, eqn $(C4')_1$ was established in full generality by Nazarov (1989) and Rice (1989).

Appendix D

The aim of this Appendix is to study the asymptotic behavior of the SIFs of a surface crack near a free surface. The notations employed are completely independent of those in the rest of the paper.

Let O denote the point of intersection of the crack front and the free surface. To the lowest approximation, the respective positions of the free surface, the crack surface and the front of the crack at the point O can be characterized by two angles, say φ_1 and φ_2. φ_1 is the angle between the tangent planes at O to the free surface and the surface of the crack, and φ_2 is the angle between the intersection of these tangent planes and the local tangent to the crack front. To the next approximation, one must specify the curvature tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ of the free surface and the surface of the crack at O, plus the curvature Γ of the projection of the crack front onto the tangent plane to the crack at O. (Again, we shall be satisfied with this degree of accuracy, because refining it would not change the conclusion to be reached in any way.)

Let the boundary of the body Ω considered be subjected to an arbitrary loading, and let us consider spheres $\mathscr{S}(O,R)$ of centre O and sufficiently small radius R for the crack to be a surface crack within them. Let $\mathscr{T}(R)$ denote the traction field exerted on $\partial\mathscr{S}(O,R)\cap\Omega$, as a result of the loading. Finally, let us consider a point on the crack front located at the distance σ, as measured along that front, from the free surface. The SIFs $K_p(\sigma)$ at that point depend on all the geometric and mechanical parameters of the problem, which can be written symbolically

$$\mathbf{K}(\sigma)\equiv(K_{\mathrm{I}},K_{\mathrm{II}},K_{\mathrm{III}})(\sigma)\equiv\mathscr{P}(R,\varphi_1,\varphi_2,\mathbf{C}_1,\mathbf{C}_2,\Gamma,\sigma;\mathscr{T}(R)) \tag{D1}$$

where the functional $\mathscr{P}$ is linear with respect to the traction field. If all distances and displacements are multiplied by $\lambda > 0$ while the stresses are kept unchanged, the SIFs are multiplied by $\sqrt{\lambda}$ as usual. It follows that

$$\mathscr{P}(\lambda R, \varphi_1, \varphi_2, \mathbf{C}_1/\lambda, \mathbf{C}_2/\lambda, \Gamma/\lambda, \lambda\sigma; \mathscr{T}) = \sqrt{\lambda}\,\mathscr{P}(R, \varphi_1, \varphi_2, \mathbf{C}_1, \mathbf{C}_2, \Gamma, \sigma; \mathscr{T}) \tag{D2}$$

for any traction field $\mathscr{T}$.

Let us assume that $\mathbf{K}(\sigma)$ behaves like σ^β for $\sigma \to 0$, where β is some exponent to be determined. This means that the functional $\mathscr{P}$ can be written, for $\sigma \to 0$, as

$$\mathscr{P}(R, \varphi_1, \varphi_2, \mathbf{C}_1, \mathbf{C}_2, \Gamma, \sigma; \mathscr{T}) = \mathscr{P}^{(\beta)}(R, \varphi_1, \varphi_2, \mathbf{C}_1, \mathbf{C}_2, \Gamma; \mathscr{T})\sigma^\beta + o(\sigma^\beta). \tag{D3}$$

Inserting this equation into the homogeneity property (D2) and identifying terms of order σ^β, one gets the following homogeneity property for the functional $\mathscr{P}^{(\beta)}$:

$$\mathscr{P}^{(\beta)}(\lambda R, \varphi_1, \varphi_2, \mathbf{C}_1/\lambda, \mathbf{C}_2/\lambda, \Gamma/\lambda; \mathscr{T}) = \lambda^{1/2-\beta}\mathscr{P}^{(\beta)}(R, \varphi_1, \varphi_2, \mathbf{C}_1, \mathbf{C}_2, \Gamma; \mathscr{T}). \tag{D4}$$

By eqns (D1), (D3) and (D4) (applied with $\lambda = 1/R$),

$$\mathbf{K}(\sigma) = \mathbf{K}^{(\beta)}\sigma^\beta + o(\sigma^\beta), \tag{D5}$$

where

$$\mathbf{K}^{(\beta)} = \mathscr{P}^{(\beta)}(R, \varphi_1, \varphi_2, \mathbf{C}_1, \mathbf{C}_2, \Gamma; \mathscr{T}(R)) = R^{1/2-\beta}\mathscr{P}^{(\beta)}(1, \varphi_1, \varphi_2, R\mathbf{C}_1, R\mathbf{C}_2, R\Gamma; \mathscr{T}(R)). \tag{D6}$$

Now it has been shown by Bazant and Estenssoro (1979) and Leguillon (1995) that near the point O, the stress field is of the form

$$\sigma_{ij}(\rho, \psi, \chi) = \kappa f_{ij}^{(\alpha)}(\psi, \chi)\rho^\alpha + o(\rho^\alpha) \tag{D7}$$

where ρ, ψ, χ denote spherical coordinates with origin at O, α an exponent depending on the angles φ_1 and φ_2, the $f_{ij}^{(\alpha)}$ some universal functions depending only on φ_1 and φ_2 (and Poisson's ratio), and κ some scalar depending on the geometry and the loading.[11] It follows that the traction on $\partial\mathscr{S}(s, R) \cap \Omega$ is of the form

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} \equiv \kappa\mathbf{f}^{(\alpha)}(\psi, \chi)R^\alpha + o(R^\alpha).$$

Inserting this expression into eqn (D6), we get

$$\begin{aligned}\mathbf{K}^{(\beta)} &= R^{\alpha+1/2-\beta}\kappa\mathscr{P}^{(\beta)}[1, \varphi_1, \varphi_2, R\mathbf{C}_1, R\mathbf{C}_2, R\Gamma; \{\mathbf{f}^{(\alpha)}(\psi, \chi)\}] + o(R^{\alpha+1/2-\beta}) \\ &= R^{\alpha+(1/2)-\beta}\kappa\mathscr{P}^{(\beta)}[1, \varphi_1, \varphi_2, \mathbf{0}, \mathbf{0}, 0; \{\mathbf{f}^{(\alpha)}(\psi, \chi)\}] + o(R^{\alpha+1/2-\beta}).\end{aligned} \tag{D8}$$

This equation holds for all sufficiently small values of R. But $\mathbf{K}^{(\beta)}$ is, by definition, independent of R; thus, the exponent $\alpha + \frac{1}{2} - \beta$ must be zero, which means that

$$\beta = \alpha + \tfrac{1}{2}. \tag{D9}$$

[11] In fact, there are three modes and not only one as eqn (D7) seems to suggest; but these modes correspond to different exponents α in general, so that one of them predominates.

Now the exponents α found by Bazant and Estenssoro (1979) and Leguillon (1995) are all greater than $-3/2$.[12] It then follows from eqn (D9) that β is always greater than -1, as announced in Appendix C.

Remark. Taking the limit $R \to 0$ in eqn (D8), one gets

$$\mathbf{K}^{(\beta)} = \kappa\mathscr{P}^{(\beta)}[1, \varphi_1, \varphi_2, \mathbf{0}, \mathbf{0}, 0; \{\mathbf{f}^{(\alpha)}(\psi, \chi)\}]$$

which shows that $\mathbf{K}^{(\beta)}$ is a universal function of φ_1, φ_2 and κ (linear with respect to the last argument).

Appendix E

The goal of this Appendix is to establish the universality of the double limit

$$\lim_{\sigma\to 0}\lim_{R\to 0}\int_{s-\sigma}^{s+\sigma} \mathbf{Z}(\Omega, R, s, s')(s'-s)\,\mathrm{d}s.$$

This will be done, like for the proof of the universality of $\lim_{s'\to s}\mathbf{Z}(\Omega, s, s')(s'-s)^2$ (see Appendix B), by replacing the operator $\mathbf{Z}(\Omega, R, s, s')$ by the operator $\mathbf{Z}(R_0, R, s, s')$ and letting R_0 go to zero using some homogeneity property for $\mathbf{Z}(R_0, R, s, s')$.

One has

$$\lim_{\sigma\to 0}\lim_{R\to 0}\int_{s-\sigma}^{s+\sigma} [\mathbf{Z}(\Omega, R, s, s')-\mathbf{Z}(R_0, R, s, s')](s'-s)\,\mathrm{d}s = 0$$

because we have seen in Appendix B that the difference $\mathbf{Z}(\Omega, R, s, s')-\mathbf{Z}(R_0, R, s, s')$ is independent of R, continuous with respect to s' and finite for $s' = s$. Thus,

$$\lim_{\sigma\to 0}\lim_{R\to 0}\int_{s-\sigma}^{s+\sigma} (\Omega, R, s, s')(s'-s)\,\mathrm{d}s = \lim_{\sigma\to 0}\lim_{R\to 0}\int_{s-\sigma}^{s+\sigma} \mathbf{Z}(R_0, R, s, s')(s'-s)\,\mathrm{d}s. \tag{E1}$$

Let us put

$$\int_{s-\sigma}^{s+\sigma} \mathbf{Z}(R_0, R, s, s')(s'-s)\,\mathrm{d}s' \equiv \mathscr{Q}(R_0, \sigma, R, \mathbf{C}, \Gamma) \tag{E2}$$

and

$$\mathscr{R}(R_0, \mathbf{C}, \Gamma) \equiv \lim_{\sigma\to 0}\lim_{R\to 0}\mathscr{Q}(R_0, \sigma, R, \mathbf{C}, \Gamma). \tag{E3}$$

It is easy to check that just like $\mathbf{Z}(R_0, s, s')$ (see Appendix B), $\mathbf{Z}(R_0, R, s, s')$ is positively homogeneous of degree -2. It then follows from eqn (E2) that for every positive λ,

[12] This is not surprising, since in three dimensions, $\alpha > -3/2$ is the condition which ensures that the integral expressing the total elastic energy is convergent.

$$\mathcal{Q}(\lambda R_0, \lambda\sigma, \lambda R, \mathbf{C}/\lambda, \Gamma/\lambda) = \mathcal{Q}(R_0, \sigma, R, \mathbf{C}, \Gamma).$$

Taking the double limit $R \to 0$, then $\sigma \to 0$ in this equation, we get

$$\mathcal{R}(\lambda R_0, \mathbf{C}/\lambda, \Gamma/\lambda) = \mathcal{R}(R_0, \mathbf{C}, \Gamma)$$

which yields, with $\lambda = 1/R_0$ and in the limit $R_0 \to 0$:

$$\mathcal{R}(R_0, \mathbf{C}, \Gamma) = \mathcal{R}(1, \mathbf{0}, 0).$$

This result, combined with eqn (E1), establishes the universality of the double limit in question.

Appendix F

In this Appendix, we provide the values of the non-zero components of the universal operators $\mathbf{M}(\varphi)$, $\mathbf{N}(\varphi)$, $\mathbf{P}(\varphi)$, and also those of the non-universal operator $\mathbf{Z}(\Omega, s, s')$ for the important special case of a semi-infinite plane crack in an infinite body.

Table F1 gives the numerical values of the "in-plane" components of the operator $\mathbf{M}(\varphi)$ for $0 \leqslant \varphi \leqslant 80^\circ$. Values for $\varphi < 0$ can be obtained by using the easily proved fact that $M_{\mathrm{I,I}}$ and $M_{\mathrm{II,II}}$ are odd, and $M_{\mathrm{I,II}}$ and $M_{\mathrm{II,I}}$ even, functions of φ. The "out-of-plane" component $M_{\mathrm{III,III}}(\varphi)$ is given by the following formula:

$$M_{\mathrm{III,III}}(\varphi) = \frac{1}{2\sin(\pi m)}\left[-\left(\frac{1-m}{1+m}\right)^{m/2}\cos(\pi m) + \frac{1-7m^2}{1-m^2}\left(\frac{1-m}{1+m}\right)^{3m/2}\right], \quad m \equiv \frac{\varphi}{\pi}. \tag{F1}$$

Table F1
Numerical values of the components of the in-plane operator $\mathbf{M}(\varphi)$

φ (°)	$M_{\mathrm{I,I}}(\varphi)$	$M_{\mathrm{I,II}}(\varphi)$	$M_{\mathrm{II,I}}(\varphi)$	$M_{\mathrm{II,II}}(\varphi)$
0	0	−1.500	0.500	0
5	−0.065	−1.491	0.497	−0.119
10	−0.130	−1.464	0.488	−0.236
15	−0.192	−1.420	0.474	−0.349
20	−0.252	−1.358	0.454	−0.457
25	−0.307	−1.282	0.429	−0.556
30	−0.359	−1.192	0.399	−0.645
35	−0.405	−1.089	0.366	−0.724
40	−0.446	−0.975	0.328	−0.791
45	−0.481	−0.853	0.289	−0.845
50	−0.510	−0.724	0.247	−0.885
55	−0.532	−0.590	0.205	−0.911
60	−0.548	−0.454	0.162	−0.924
65	−0.557	−0.318	0.119	−0.922
70	−0.560	−0.184	0.077	−0.907
75	−0.557	−0.053	0.037	−0.879
80	−0.548	0.073	−0.001	−0.840

Table F2 provides the numerical values of the non-zero components of the operator $\mathbf{N}(\varphi)$ for the value $\nu = 0.3$ and for $0 \leqslant \varphi \leqslant 80^\circ$; $N_{\mathrm{I,III}}$ and $N_{\mathrm{III,I}}$ are odd, and $N_{\mathrm{II,III}}$ and $N_{\mathrm{III,II}}$ even, functions of φ.

Table F3 provides the numerical values of the non-zero components of the operator $\mathbf{P}(\varphi)$ in a similar way, again for $\nu = 0.3$. $P_{\mathrm{I,III}}$ and $P_{\mathrm{III,I}}$ are even, and $P_{\mathrm{II,III}}$ and $P_{\mathrm{III,II}}$ odd, functions of φ. It is also possible to calculate the components of the operator $\mathbf{P}(0)$ in a completely explicit and analytical way, and the results read as follows:

$$P_{\mathrm{I,III}}(0) = -2; \quad P_{\mathrm{II,III}}(0) = 0; \quad P_{\mathrm{III,I}}(0) = \frac{2(1-\nu)^2}{2-\nu}; \quad P_{\mathrm{III,II}}(0) = 0. \tag{F2}$$

A comparison with the works of Gao (1992), Xu et al. (1994) and Ball and Larralde (1995) is in order here. Although, as explained in the Introduction of Part I, these works did not rely on the same principle as the present one, and in particular did not identify the universal operator $\mathbf{P}(\varphi)$ as such, it is possible to establish some connection with them. It is found that Ball and Larralde's result for K_{III} along the extended crack front is compatible with Mouchrif's (1994) value for $P_{\mathrm{III,I}}(0)$ given by eqn $(\mathrm{F2})_3$, whereas those of Gao and Xu et al. are not and furthermore differ from each other. The true value of $P_{\mathrm{III,I}}(0)$ is therefore thought to be that of Mouchrif and Ball and Larralde.

Finally, the operator $\mathbf{Z}(\Omega, s, s')$ is diagonal for a semi-infinite plane crack in an infinite body, and its diagonal components are given by

Table F2
Numerical values of the components of the operator $\mathbf{N}(\varphi)$

φ (°)	$N_{\mathrm{I,III}}(\varphi)$	$N_{\mathrm{II,III}}(\varphi)$	$N_{\mathrm{III,I}}(\varphi)$	$N_{\mathrm{III,II}}(\varphi)$
0	0	−1.176	0	0.824
5	−0.021	−1.176	0.050	0.817
10	−0.041	−1.175	0.100	0.798
15	−0.061	−1.173	0.148	0.767
20	−0.080	−1.170	0.193	0.724
25	−0.098	−1.166	0.236	0.670
30	−0.115	−1.161	0.276	0.606
35	−0.130	−1.155	0.311	0.534
40	−0.144	−1.147	0.342	0.454
45	−0.157	−1.137	0.368	0.368
50	−0.169	−1.125	0.390	0.278
55	−0.178	−1.112	0.406	0.185
60	−0.187	−1.096	0.417	0.090
65	−0.194	−1.077	0.424	−0.004
70	−0.200	−1.056	0.425	−0.098
75	−0.205	−1.033	0.422	−0.188
80	−0.209	−1.007	0.415	−0.275

Table F3
Numerical values of the components of the operator $\mathbf{P}(\varphi)$

φ (°)	$P_{\mathrm{I,III}}(\varphi)$	$P_{\mathrm{II,III}}(\varphi)$	$P_{\mathrm{III,I}}(\varphi)$	$P_{\mathrm{III,II}}(\varphi)$
0	−2.000	0	0.576	0
5	−1.998	−0.013	0.574	−0.095
10	−1.992	−0.025	0.565	−0.189
15	−1.982	−0.037	0.552	−0.280
20	−1.968	−0.048	0.533	−0.366
25	−1.951	−0.059	0.510	−0.445
30	−1.930	−0.068	0.483	−0.517
35	−1.906	−0.077	0.452	−0.581
40	−1.878	−0.084	0.418	−0.634
45	−1.848	−0.090	0.381	−0.678
50	−1.816	−0.095	0.343	−0.711
55	−1.780	−0.099	0.303	−0.733
60	−1.743	−0.101	0.263	−0.743
65	−1.704	−0.103	0.224	−0.743
70	−1.663	−0.104	0.185	−0.731
75	−1.620	−0.104	0.147	−0.710
80	−1.576	−0.103	0.110	−0.679

$$Z_{\mathrm{I,I}}(\Omega,s,s') = \frac{1}{2\pi}\frac{1}{(s'-s)^2};\quad Z_{\mathrm{II,II}}(\Omega,s,s') = \frac{1}{2\pi}\frac{2-3\nu}{2-\nu}\frac{1}{(s'-s)^2};$$
$$Z_{\mathrm{III,III}}(\Omega,s,s') = \frac{1}{2\pi}\frac{2+\nu}{2-\nu}\frac{1}{(s'-s)^2}. \tag{F3}$$

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Glossary

Definition of notations used in appendices A, B, C and E

For ease of reference, we summarize here some important notations used in all Appendices except Appendices D and F.

$K_p^{(i)}(\Omega, s', M)$: p-th SIF created at the point s' of the initial crack front by a unit point force exerted on the point M in the direction $\mathbf{E}_i \equiv \partial \mathcal{O}\mathbf{M}/\partial X_i$, $\partial\Omega_u$ being simultaneously clamped and $\partial\Omega_t$ free of tractions.

$K_p^{(i)}(\Omega, s, s')$: p-th SIF created at the point s of the initial crack front by a unit "doublet" exerted on the point s' of that front in the direction $\mathbf{E}_i$, $\partial\Omega_u$ being clamped and $\partial\Omega_t$ free of tractions.

$K_p^{(i)}(R, s, s')$: p-th SIF created at the point s of the initial crack front by a unit "doublet" exerted on the point $s'(\in \mathcal{S}(s, R))$ of that front in the direction $\mathbf{E}_i$, the boundary of the sphere $\mathcal{S}(s, R)$ of centre s and radius R being free of tractions.

$\mathbf{L}_M$: differential operator which evaluates the stresses at the point $M(\in \partial\mathcal{S}(s, R))$ from the displacements, then the traction $\mathbf{t}(M) \equiv \boldsymbol{\sigma}(M) \cdot \mathbf{n}(M)$.

$v_i^{(j)}(\Omega, M, s')$: i-th component of the displacement at the point M generated by a unit doublet exerted on the point s' of the initial crack front in the direction $\mathbf{E}_j$, $\partial\Omega_u$ being clamped and $\partial\Omega_t$ free of tractions.

$\mathbf{Z}(\Omega, R, s, s')$: up to some multiplicative factors, the components $Z_{pq}(\Omega, R, s, s')$ represent SIFs at the point s of the initial crack front created by the application, on the boundary of the sphere $\mathscr{S}(s, R)$, of traction fields of the form $\{\mathbf{L}_M \cdot \mathbf{v}^{(j)}(\Omega, M, s')\}$ arising themselves from application of unit doublets on the point s' of the initial crack front, $\partial\Omega_u$ being clamped and $\partial\Omega_t$ free of tractions [see eqn (A2)].

$\mathbf{Z}(R_0, R, s, s')$ (with $R_0 > R$): same quantity as $\mathbf{Z}(\Omega, R, s, s')$, the body Ω being replaced by the sphere $\mathscr{S}(s, R_0)$ with traction-free boundary.

$\mathbf{Z}(\Omega, s, s')$: constant value of $\mathbf{Z}(\Omega, R, s, s')$ for $R < D(s, s')$ (see Lemma 1).

$\mathbf{Z}(R_0, s, s')$: constant value of $\mathbf{Z}(R_0, R, s, s')$ for $R < D(s, s')$.

Let us stress the consistency of these notations. For functions of three variables, the first argument (Ω, R or R_0) underlines the dependence upon the geometry of the domain considered; the second one indicates the point of observation of the function, and the third one the point of application of the loading. For functions of four variables, an extra argument indicating a dependence upon some other geometric parameter is inserted after the first one.